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Antonija Horvatek

UROŠ MILUTINOVIĆ–DARKO ŽUBRINIĆ

BALKANIAN MATHEMATICAL OLYMPIADES

1984 – 1991

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ZAGREB, 1991

Introduction

The organization of Balkan mathematical olympiads was initiated by Prof. Dimitrios Kontogiannis froma Athens (Greece) and Prof. Ivan Tonov from Sofia (Bulgaria) in a friendly chat during the 24th IMO in Paris, 1983. The first competition was held in Athens in 1984. The main objective of Balkan olympiads is to develop friendly relations among Balkan countries (Greece, Cyprus, Bulgaria, Romania and Yugoslavia; we hope that Albania and Turkey will soon join the competition) and to train national teams for the International mathematical olympiad. Let us mention that Yugoslavia participates this competition from the fourth Olympiade.

The competition is held every year in May. Each team has six pupils plus three leaders. Problems are selected by the international jury just before the contest. Duration of the exam is 4,5 hours during which pupils solve four problems. Each problem is worth 10 points, although they are not of equal difficulty. As a rule problems are listed with incresing difficulty. Up to now there were only two problems that were not completely solved during the competiton (problems no. 6.2. and 6.4.).

We decided to present also some of the proposals that have not been selected by the jury. Majority of problems in this book are published for the first time.

Let us mention that in 1989 Balkan summer school was intiated (for younger competitors). It is held usually in some tourist resort at the coast. Every participanting country is represented by ten students and three leaders, who are also supposed to be lecturers. The official language is English.

In the course of past ten years there appeared many new regional competitions, and this trend is certainly going to continue. Let us mention a competition among Austria and Poland, initiated by the cultural agreement between the two states, Ibero–American competition, (South American countries and Spain), Maghrebian competition of north–african countries, Nordic competition and since recently Asian–Pacific competition.

The numbering of figures in this book is related to corresponding problems.

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Uroš Milutinović and Darko Žubrinić

NOTATIONS

- $\mathbf{N} = \{1, 2, 3, \dots\}$, the set of positive integers
- $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$
- $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ sets of integers, rational and real numbers
- Euler's function $\varphi(n)$: the number of positive integers not greater than n and relatively prime to $n \in \mathbf{N}$.
- $\lfloor x \rfloor$ the greatest integral part of $x \in \mathbf{R}$ (i.e. the greatest integer not exceeding x).
- iff $\equiv$ if and only if
- We shall adopt the notation (a, b) for both the ordered pair and the open interval, which will be easy to destinguish from the context. Closed intervals will be denoted by $[a, b]$.
- $|S|$ =cardinality of the set S.

1st BALKANIAN MATHEMATICAL OLYMPIAD

ATHENS, Greece, 1984

1.1. Let $a_1, a_2, \dots, a_n$ be positive real numbers ($n \geq 2$) such that $a_1+a_2+\cdots+a_n = 1$. Prove that

$$\frac{a_1}{1+a_2+a_3+\cdots+a_n} + \frac{a_2}{1+a_1+a_3+\cdots+a_n} + \dots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}} \geq \frac{n}{2n-1}.$$

(Greece)

1.2. Let $ABCD$ be inscribed quadrelateral and H_A, H_B, H_C, H_D intersections of altitudes of triangles BCD, CDA, DAB and ABC respectively. Prove that quadrelaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.

(Romania)

1.3. Prove that for every natural number m there exists n, $n > m$, such that the decimal representation of 5^n is obtained from the decimal representation of 5^m by adding a certain number of digits to the left.

(Bulgaria)

1.4. Find all real solutions of the system

$$\begin{aligned} ax + by &= (x-y)^2 \\ by + cz &= (y-z)^2 \\ cz + ax &= (z-x)^2, \end{aligned}$$

where a, b, c are given positive real numbers.

(Romania)

2nd BALKANIAN MATHEMATICAL OLYMPIAD

SOFIA, Bulgaria, May 1985.

2.1. Let O be the circumcenter of a triangle ABC, D the midpoint of AB and E the barycenter of the triangle ACD. Show that $CD \perp OE$ if and only if $AB = AC$.

(Bulgaria)

2.2. Let $a, b, c, d \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be such that

$$\sin a + \sin b + \sin c + \sin d = 1$$

and

$$\cos 2a + \cos 2b + \cos 2c + \cos 2d \geq \frac{10}{3}.$$

Show that $a, b, c, d \in [0, \frac{\pi}{6}]$.

(Romania)

2.3. The real points of the form $19a + 85b$, where $a, b \in \mathbf{N}_0$, are coloured red, while all the rest of the integral points of $\mathbf{R}$ are coloured green. Examine whether there is a point $A \in \mathbf{R}$ such that for every pair $(B, C) \in \mathbf{Z} \times \mathbf{Z}$ with B, C symmetrical with respect to A, the colours of B and C are different.

(Greece)

2.4. 1985 people participate a conference. In each group of three people there are at least two speaking a common language. If each person speaks at most five languages, show that there are at least 200 persons on this conference speaking a common language.

(Romania)

3rd BALKANIAN MATHEMATICAL OLYMPIAD

BUCHAREST, Romania, May 1986.

3.1. A straight line passing through the center I of the incircle of the triangle ABC intersects its circumscribed circle in points F and G, and the incircle in points D and E, where D lies between I and F.
Prove that $DF \cdot EG \geq r^2$, where r is the radius of the incircle. When does the equality hold?

(Greece)

3.2. Let $ABCD$ be a thetraedron and E, F, G, H, K, L points lying on AB, BC, CA, DA, BD DC respectively.
Prove that if $AE{\cdot}BE = BF{\cdot}CF = CG{\cdot}AG = DH{\cdot}AH = DK{\cdot}BK = DL{\cdot}CL$, then points E, F, G, H, K, L lie on a sphere.

(Bulgaria)

3.3. The sequence $a_1, a_2, \dots$ is defined by $a_1 = a$, $a_2 = b$ and $a_{n+1} = (a_n^2 + c)/a_{n-1}$ for $n = 2, 3, \dots$, where a, b, c are real numbers, such that $ab \neq 0$, $c > 0$.
Prove that a_n $(n = 1, 2, \dots)$ are integers if and only if a, b and $(a^2 + b^2 + c)/ab$ are integers.

(Bulgaria)

3.4. A triangle ABC and a point T lie in a plane, so that the triangles TAB, TBC, TCA have the same circumference and the same area. Prove that

a) if T is in the interior of the triangle ABC, then ABC is equilateral,

b) if T is not in the interior of the triangle ABC, then ABC is rectangular.

(Romania)

4th BALKANIAN MATHEMATICAL OLYMPIAD

ATHENS, Greece, May 1987.

4.1. Let a be a real number and $f\colon \mathbf{R} \to \mathbf{R}$ a function such that for all $x, y \in \mathbf{R}$

$$f(x+y) = f(x)f(a-y) + f(y)f(a-x)$$
$$f(0) = \frac{1}{2}$$

Prove that f is a constant function.

(former Yugoslavia)

4.2. Let $x \geq 1$ and $y \geq 1$ be such that

$$a = \sqrt{x-1} + \sqrt{y-1}$$
$$b = \sqrt{x+1} + \sqrt{y+1}$$

are nonconsecutive integers. Prove that $b = a + 2$ and $x = y = \frac{5}{4}$.

(Romania)

4.3. In a triangle ABC the following relation holds:

$$\sin^{23}\frac{\alpha}{2} \cdot \cos^{48}\frac{\beta}{2} = \sin^{23}\frac{\beta}{2} \cdot \cos^{48}\frac{\alpha}{2},$$

where α and β are the corresponding angles at A and B. Find the ratio AC/BC.

(Cyprus)

4.4. Two circles k_1 and k_2 with centers at O_1 and O_2 with radii 1 and $\sqrt{2}$ intersect in two points A and B, and $O_1O_2 = 2$. Let AC be a chord on k_2. Find the length of AC, if the midpoint of AC lies on k_1.

(Bulgaria)

5th BALKANIAN MATHEMATICAL OLYMPIAD

NICOSIA, Cyprus, May 1988.

5.1. Let CH, CL and CM be altitude, bisector and a median of the triangle ABC with points H, L and M lying on AB.
The ratio of areas of triangles HMC and ABC is $\frac{1}{4}$, while the corresponding ratio for triangles LMC and ABC is $1-\frac{\sqrt{3}}{2}$.
Determine the angles of the triangle ABC.

(Bulgaria)

5.2. Find all polynomials $P(x,y)$ in two variables, such that

$$P(a,b)\cdot P(c,d)=P(ac+bd,ad+bc)$$

for all real numbers a, b, c, d.

(former Yugoslavia)

5.3. Prove that every thetraedron $A_1A_2A_3A_4$ can be situated between two parallel planes whose distance is not greater than $\frac{1}{2}\sqrt{P/3}$, where

$$P=(A_1A_2)^2+(A_1A_3)^2+(A_1A_4)^2+(A_2A_3)^2+(A_2A_4)^2+(A_3A_4)^2.$$

(Greece)

5.4. Find all pairs a_n, a_{n+1} of consecutive members of the sequence $a_1,a_2\dots,$ defined by $a_n=2^n+49$, so that

$$a_n=p\cdot q,\quad a_{n+1}=r\cdot s$$

where p, q, r, s are prime numbers such that

$$p<q,\quad r<s,\quad q-p=s-r.$$

(Romania)

6th BALKANIAN MATHEMATICAL OLYMPIAD

SPLIT, former Yugoslavia, May 1989.

6.1. Let $d_1, d_2, \ldots, d_k$ be all the divisors of a positive integer n and let $1 = d_1 < d_2 < \cdots < d_k = n$. Find all numbers n for which $k \geq 4$ and

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 = n.$$

(Bulgaria)

6.2. Let $\overline{a_n a_{n-1} \ldots a_1 a_0} = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + 10a_1 + a_0$ be the decimal representation of a prime number. Assuming that $n > 1$ and $a_n > 1$, show that the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is irreducible, i.e. it cannot be the product of two polynomials of positive degree and with integral coefficients.

(former Yugoslavia)

6.3. Let ABC be a triangle and let ℓ be a straight line intersecting the sides AB and AC at the points B_1 and C_1 respectively, so that the vertex A and the barycentre G of ABC lie on the same half–plane defined by the line ℓ. Show that

$$\text{Area}(BB_1GC_1) + \text{Area}(CC_1GB_1) \geq \frac{4}{9}\text{Area}(ABC).$$

When does the equality hold?

(Greece)

6.4. We consider families $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$, $(n \geq 3)$, such that:

(i) If $A \in \mathcal{F}$, then $|A| = 3$;

(ii) If $A \in \mathcal{F}$, $B \in \mathcal{F}$, $A \neq B$, then $|A \cap B| \leq 1$.

Let $f(n)$ be the maximum value of $|\mathcal{F}|$ for all such families $\mathcal{F}$. Show that

$$\frac{1}{6}(n^2 - 4n) \leq f(n) \leq \frac{1}{6}(n^2 - n),$$

($|S|$ denotes the cardinality of a set S).

(Romania)

7th BALKANIAN MATHEMATICAL OLYMPIAD

SOFIA, Bulgaria, May 1990.

7.1. Let $a_1 = 1$, $a_2 = 3$ and $a_{n+2} = (n+3)a_{n+1} - (n+2)a_n$ for every integer $n \geq 1$. Find all values of n for which 11 divides a_n.

(Greece)

7.2. Consider the polynomial defined by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n} = (1 \cdot x + 2 \cdot x^2 + \cdots + n \cdot x^n)^2.$$

Show that

$$a_{n+1} + a_{n+2} + \cdots + a_{2n} = \frac{n(n+1)(5n^2+5n+2)}{24}.$$

(Bulgaria)

7.3. Let $A_1B_1C_1$ be the orthocentric triangle of sharp–angled, non–equilateral triangle ABC. Let A_2, B_2, C_2 be points where the inscribed circle to the triangle $A_1B_1C_1$ meets its sides. Prove that Euler's lines of triangles ABC and $A_2B_2C_2$ coincide.
Remark:
(I) Vertices of orthocentric triangle are feet of heights of the given triangle.
(II) Euler's line of the given triangle ABC is by definition determined by its orthocenter and the center of the incircle.

(former Yugoslavia)

7.4. Find the minimal cardinality of a finite set A, for which there is a function $f\colon \mathbf{N} \to A$ such that if the number $|i - j|$ is prime, then $f(i) \neq f(j)$.

(Romania)

8th BALKANIAN MATHEMATICAL OLYMPIAD

CONSTANŢA, Romania, 1991.

8.1. Let M be a point on the arc AB of the circumcircle of a tringle ABC. The perpendicular to the radius OA drawn from M intersects the sides AB, AC at the points K, L respectively (O is the center of the circumcircle). Similarly, the perpendicular to the radius OB drawn from M intersects the sides AB, BC at the point N, P respectively. If $KL = MN$, compute the angle $\angle MLP$.

(Greece)

8.2. Prove that there are infinitely many noncongruent triangles T such that
- (i) the lengths a, b, c of T are relatively prime integers;
- (ii) the area of the triangle T is an integer;
- (iii) no altitude of T is an integer.

(Yugoslavia)

8.3. A regular hexagon of area H is inscribed in a convex polygon of area P (all vertices of the hexagon lye on the boundary of the polygon). Prove that $P \leq \frac{3}{2}H$. When does the equality hold?

(Bulgaria)

8.4. Prove that there exists no bijection $f\colon \mathbf{N} \to \mathbf{N}_0$ such that

$$f(mn) = f(m) + f(n) + 3f(m)f(n)$$

for all $m, n \geq 1$.

(Romania)

PROPOSALS OF PROBLEMS

Combinatorics

1. Let us 3–colour the points in the plane. Prove that there are two points at distance 1 having the same colour.

2. Let $\mathcal{F}$ be a collection of subsets of $\mathbf{N}$, no one containing another. Let $C(\mathcal{F})$ consist of all subsets M of $\mathbf{N}$ with the following properties:
(i) M intersects every member of $\mathcal{F}$;
(ii) There is no proper subset M' of M with the property (i).

Give an example of a collection $\mathcal{F}$ such that $C(\mathcal{F})$ is empty and 1989 belongs to exactly 1989 members of collection $\mathcal{F}$.

3. For an arbitrary polyhedron we denote by n_k the number of faces with exactly k vertices. Prove that at least one of the two following sentences is true:
(i) there exists k such that $n_k \geq 3$;
(ii) there exists $k \neq h$ such that $n_k \geq n_h \geq 2$.

4. Let n^2 distinct integers be given, each placed on a square of $n \times n$ chessboard ($n \geq 2$).

Show that it is possible to select n numbers, one from each row and column, so that if the number selected from any row is greater than another number in this row, then this last number is less than the number selected from its column.

Algebra

5. The sequence of functions P_n satisfies the following relations:

$$P_1(x) = x, \quad P_2(x) = x^3,$$
$$P_{n+1}(x) = \frac{P_n^3(x) - P_{n-1}(x)}{1 + P_n(x)P_{n-1}(x)}, \quad n = 2, 3, \ldots$$

Prove that all functions P_n are polynomials.

6. Solve the following system in real numbers:

$$2^{x^2+y} + 2^{x+y^2} = 8$$
$$\sqrt{x} + \sqrt{y} = 2.$$

Analysis

7. Let k be a natural number and

$$a_n = \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}, \quad n \in \mathbf{N}.$$

Prove that the sequence (a_n) is decreasing.

8. Find all continuous functions $f\colon \mathbf{R} \to \mathbf{R}$ such that

$$f(x+f(y)) = f(x) + y$$

for every $x, y \in \mathbf{R}$.

Geometry

9. Let ABC be a triangle and let CX and CY be two semilines (in the semiplane determined by the line AC and containing the point B) such that CX is parallel to AB and CX is in the interior of the angle $\angle BCY$.

A variable line through the point B meets CX in D and CY in E, while the line AD meets BC in F. Show that the lines EF pass through a fixed point.

10. Let ABC be a triangle with $BC = 2 \cdot AC - 2 \cdot AB$, and D be a point on the side BC. Prove that $\angle ABD = 2\angle ADB$ if and only if $BD = 3 \cdot CD$.

11. Three points A, B and C are given in the plane. Prove that C is a midpoint of the line–segment AB if and only if

$$PA^{1989} + PB^{1989} \geq 2 \cdot PC^{1989}$$

for every point P in the plane.

12. Let $ABCD$ be a regular triangular pyramid in which the side edges AD, BD, and CD are pairwise perpendicular. Find all points X in the interior or on the boundary of $ABCD$ for which the volume of the tetrahedron with vertices the orthogonal projections of X on faces of the pyramid $ABCD$ is maximal.

13. A convex n–gon is situated in a square with a side 1. Prove that there are three vertices A, B, C of the given n–gon, such that the area of triangle ABC does not exceed $8/n^2$.

Number theory

14. Let $n \geq 3$ be a natural number. Prove that

$$1989 \mid n^{n^{n^n}} - n^{n^n}.$$

15. Let $a_0, a_1, \ldots, a_8$ be integers for which

$$a_{n+1} = a_n^2 - a_n + 5, \quad n = 0, 1, \ldots, 7.$$

Prove that at least two of these numbers are not relatively prime.

16. Are there 1989 positive integers $a_1, a_2, \ldots, a_{1989}$ such that for every $i = 3, 4, \ldots, 1989$ we have

$$a_i + S_i = (a_i, S_i) + [a_i, S_i],$$

where $S_i = \sum_{j=1}^{i} a_j$ and $(\cdot,\cdot)$ and $[\cdot,\cdot]$ are the greatest common divisor and the lowest common multiple respectively.

17. Find all positive integers (x, y, z, n), such that

$$x^3 + y^3 + z^3 = nx^2y^2z^2.$$

18. Find all integers p for which there exist rational numbers a and b, such that the polynomial $x^5 - px - 1$ has at least one common root with a polynomial $x^2 - ax + b$.

SOLUTIONS

1.1. The inequality given in the problem is equivalent to

$$\sum_{k=1}^{n} \frac{a_k}{2-a_k} \geq \frac{n}{2n-1}.$$

Noting that

$$\frac{a_k}{2-a_k} = -1 + \frac{2}{2-a_k} \tag{1}$$

and

$$\sum_{k=1}^{n}(2-a_k) = 2n-1,$$

it is reasonable to introduce a new variable

$$x_k = \frac{2-a_k}{2n-1} > 0$$

for which

$$\sum_{k=1}^{n} x_k = 1.$$

Then substituting (1) into the given inequality, we see that it is equivalent to

$$\sum_{k=1}^{n} \frac{1}{x_k} \geq n^2. \tag{2}$$

But this is a direct consequence of the harmonic mean–arithmetic mean inequality:

$$\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}} \leq \frac{x_1+\cdots+x_n}{n} \tag{3}$$

where $x_k > 0$.

REMARK It is easy to prove (2) by induction. For $n = 1$ the statement is trivial.

Assume that (2) is true for $n-1$ and let $\sum_{k=1}^{n} x_k = 1$. Then

$$\sum_{k=1}^{n-1} \frac{x_k}{1-x_n} = 1$$

and from the inductive hypotheses we have

$$\sum_{k=1}^{n-1} \frac{1-x_n}{x_k} \geq (n-1)^2$$

i.e.

$$\sum_{k=1}^{n} \frac{1}{x_k} \geq \frac{(n-1)^2}{1-x_n} + \frac{1}{x_n} \geq n^2$$

The last inequality is verified directly.

Note that, conversly, the inequlity (3) is almost immediate consequence of (2).

1.2. The idea of the proof is to show that ABH_AH_B, BCH_BH_C, CDH_CH_D and DAH_DH_A are parallelograms. From this will follow that the quadrelaterals $ABCD$ and $H_AH_BH_CH_D$ are symmetric with respect to a point.

Figure 1.2.

It suffices to show that CDH_CH_D is a parallelogram. First we show that the quadrelateral AH_DBH_C is inscribed. Indeed, we have

$$\angle AH_DB = \angle ACB = \angle ADB = \pi - \angle BH_CA$$

The second, third and fourth equality follow from the fact that the quadrelateral $A'H_DCB''$ is inscribed (note that $\angle CB''H_D = \angle CA'H_D$), as well as $ABCD$ and $DB'H_CA$". This proves that AH_DBH_C is inscribed.

Therefore we have

$$\begin{aligned}\angle H_DH_CD &= \angle BH_CD + \angle BH_CD \\ &= (\pi - \angle BAD) + \angle BAH_D \\ &= \angle DCB + \angle BCH_D = \angle DCH_D\end{aligned}$$

where in the second line we used the fact that the quadrelateral $AD'H_CB'$ is inscribed.

In a similar way we get

$$\begin{aligned}\angle H_CH_DC &= \angle AH_DC - \angle AH_DH_C \\ &= (\pi - \angle ABC) - \angle ABB' \\ &= \angle ADC - \angle ADD' = \angle H_CDC\end{aligned}$$

Therefore CDH_CH_D is a parallelogram. The degenerated case where, say, the angles $\angle ADC$ and $\angle ABC$ are $\pi/2$ (and so $B = H_D$), can be treated in a more direct way:

$$\begin{aligned}&\angle BH_CD = \angle B'H_CD' = \pi - \angle B'AD' = \angle DCB \\ = \angle H_CDC = \angle ADC - \angle ADD' &= \angle ABC - \angle ABB' = \angle H_CBC.\end{aligned}$$

1.3. If $n > m$, then the statement in the problem is equivalent to

$$\begin{aligned}5^m &= \overline{a_{k_m} \dots a_0} \\ 5^n &= \overline{a_{k_n} \dots a_{k_m} \dots a_0},\end{aligned}$$

where the right–hand sides represent the corresponding decimal representations. This is equivalent to

(1) $$5^n - 5^m \equiv 0 \pmod{10^{k+1}}$$

with $k = k_m$. It is clear that $k + 1 \leq m$, because $5^m = a_k \cdot 10^k + \dots,\ a_k \geq 1$. Therefore $5^n - 5^m$ is divisible by 5^{k+1} for any $n > m$. In view of (1), it is only left to find n, $n > m$, such that $5^n - 5^m$ is divisible by 2^{k+1}, or equivalently

$$5^{n-m} - 1 \equiv 0 \pmod{2^{k+1}}.$$

But by the Euler formula we have

$$5^{\varphi(2^{k+1})} - 1 \equiv 0 \pmod{2^{k+1}}$$

(note that 5 and 2^{k+1} are relatively prime). So with $n = m + \varphi(2^{k+1})$ we are done.

1.4. Let us introduce new variables

$$s = x - y, \quad t = y - z,$$

Then the system becomes

$$\begin{aligned} a(s+y) + by &= s^2 \\ by + c(y-t) &= t^2 \\ c(y-t) + a(s+y) &= (s+t)^2 = s^2 + 2st + t^2 \\ &= 2st + [a(s+y) + by] + [by + c(y-t)] \end{aligned}$$

The last relation gives $y = -st/b$. After substituting, from the first two equations we obtain the following system in s and t:

$$\begin{aligned} [ab - (a+b)t]s &= s^2 \\ [-(b+c)s - bc]t &= t^2 \end{aligned}$$

We have three possibilities.
a) If $s = 0$, then $-ct = t^2$ and we have either $t = 0$, i.e.

$$(x, y, z) = (0, 0, 0),$$

or $t = -c$, i.e.

$$(x, y, z) = (0, 0, c);$$

b) In a similar way, for $t = 0$ we obtain $s = a$ and:

$$(x, y, z) = (a, 0, 0)$$

c) If $s \neq 0$ and $t \neq 0$, then

$$\begin{aligned} ab - (a+b)t &= bs \\ -bc - (b+c)s &= bt. \end{aligned}$$

As $a, b, c > 0$, the determinant of this system is different from 0. Solving it for instance by elimination, we get $s = -b$, $t = b$ and one more solution:

$$(x, y, z) = (0, b, 0).$$

Figure 2.1.

2.1. It is clear that $\overrightarrow{OD} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$ and

$$\overrightarrow{OE} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OD} + \overrightarrow{OC}) = \frac{1}{6}(3\overrightarrow{OA} + \overrightarrow{OB} + 2\overrightarrow{OC}).$$

We also have

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \frac{1}{2}[\overrightarrow{OA} + \overrightarrow{OB} - 2\overrightarrow{OC}].$$

After a short computation, from $\overrightarrow{OE} \cdot \overrightarrow{CD} = 0$ (note that $OA = OB = OC$) we get $\overrightarrow{OA} \cdot \overrightarrow{OB} = \overrightarrow{OA} \cdot \overrightarrow{OC}$, i.e. $\overrightarrow{OA} \perp \overrightarrow{BC}$. But this is easily seen to be equivalent to $AB = AC$.

2.2. Let us introduce the following notation:

$$x = \sin a, \quad y = \sin b, \quad z = \sin c, \quad u = \sin d.$$

In view of the symmetry it suffices to prove that $x \in [0, \frac{1}{2}]$. From $\cos 2a = 1 - 2x^2$, $\cos 2b = 1 - 2y^2$, $\cos c = 1 - 2z^2$ and $\cos 2d = 1 - 2u^2$ we deduce:

$$x + y + z + u = 1$$
$$x^2 + y^2 + z^2 + u^2 \leq \frac{1}{3}.$$

Eliminating u in the inequality, after a short computation we arrive to the following quadratic inequality in z:

$$z^2 + (x + y - 1)z + (x^2 + y^2 + xy - x - y + \frac{1}{3}) \leq 0.$$

In order to have a solution z, the corresponding discriminant must be nonnegative, that is

$$(x+y-1)^2 - 4(x^2+y^2+xy-x-y+\frac{1}{3}) \geq 0.$$

But this yields another quadratic inequality

$$3y^2 + 2(x-1)y + (3x^2 - 2x + \frac{1}{3}) \leq 0$$

in the variable y. For the analogous reason as before, its discriminant must be nonnegative:

$$4(x-1)^2 - 4[9x^2 - 6x + 1] \geq 0,$$

i.e.

$$x(x - \frac{1}{2}) \leq 0.$$

From this we immediately conclude that $x \in [0, \frac{1}{2}]$.

2.3. The general solution of the Diophant equation

$$19x + 85y = n \tag{1}$$

is

$$x = 19t - 2n, \quad y = 9n - 85t, \quad t \in \mathbf{Z}.$$

We shall use the fact that $1 = 9 \cdot 19 - 2 \cdot 85$, i.e. $n = 9n \cdot 19 - 2n \cdot 85$.

The condition $x, y \geq 0$ is equivalent to

$$\frac{2n}{19} \leq t \leq \frac{9n}{85}.$$

Therefore we see that the point $n \geq 0$ will be coloured red if and only if the interval

$$[\frac{2n}{19}, \frac{9n}{85}] \tag{2}$$

contains an integer. By the way, it is clear that all negative integers are coloured green.

We check directly that all integers ≥ 1512 are coloured red. Namely, by using

$$1 = 9 \cdot 19 - 2 \cdot 85 \tag{3}$$

$$1 = (-76) \cdot 19 + 17 \cdot 85 \tag{4}$$

we have

$$\begin{aligned}
1512 &= 8 \cdot 19 + 16 \cdot 85\\
1513 &= 17 \cdot 19 + 14 \cdot 85\\
1514 &= 26 \cdot 19 + 12 \cdot 85\\
1515 &= 35 \cdot 19 + 10 \cdot 85\\
1516 &= 44 \cdot 19 + 8 \cdot 85\\
1517 &= 53 \cdot 19 + 6 \cdot 85\\
1518 &= 62 \cdot 19 + 4 \cdot 85\\
1519 &= 71 \cdot 19 + 2 \cdot 85\\
\boxed{1520} &= 80 \cdot 19 + 0 \cdot 85\\
1511 &= 4 \cdot 19 + 17 \cdot 85\\
1522 &= 13 \cdot 19 + 15 \cdot 85\\
1523 &= 22 \cdot 19 + 13 \cdot 85\\
1524 &= 31 \cdot 19 + 11 \cdot 85\\
1525 &= 40 \cdot 19 + 9 \cdot 85\\
1526 &= 49 \cdot 19 + 7 \cdot 85\\
1527 &= 58 \cdot 19 + 5 \cdot 85\\
1528 &= 67 \cdot 19 + 3 \cdot 85\\
1529 &= 76 \cdot 19 + 1 \cdot 85\\
\boxed{1530} &= 0 \cdot 19 + 18 \cdot 85\\
1531 &= 8 \cdot 19 + 16 \cdot 85
\end{aligned}$$

and so on (note that (4) is used only at boxed numbers). In this way we see that all integers $n \geq 1512$ are red.

For $n = 1511$ the corresponding interval (2) is $[159\frac{1}{19}, 159\frac{84}{85}]$, which contains no integer, and is therefore coloured green.

So, if there is A with the property described in the problem, then it must be the midpoint of the interval $[0, 1511]$, i.e. $A = 755\frac{1}{2}$.

It is only left to prove that for any integer $n \in [0, 1511]$ the integers n and $1511 - n$ have different colours (they are symmetric with respect to A).

a) Let us prove that for any integer $n \in [0, 1511]$ which is red, the integer $1511 - n$ must be green.

Suppose by contradiction that they are both red, i.e.

$$\begin{aligned}
n &= 19a + 85b\\
1511 - n &= 19c - 85d
\end{aligned}$$

with $a, b, c, d \geq 0$. Then we would have the integer

$$1511 = 19(a + c) + 85(b + d)$$

coloured red, a contradiction (note that $a+c, b+d \geq 0$).

b) If n is coloured green, then

$$\frac{2n}{19}, \frac{9n}{85} \in (k, k+1)$$

for some $k \in \mathbf{Z}$, so that

$$\begin{aligned} \frac{2n}{19} &= k + \frac{\alpha}{19}, \quad \alpha \in \{1, \ldots, 18\}, \\ \frac{2n}{85} &= k + \frac{\beta}{85}, \quad \beta \in \{1, \ldots, 84\}. \end{aligned}$$

Now we have

$$\begin{aligned} \frac{2(1511-n)}{19} &= 159\frac{1}{19} - k - \frac{\alpha}{19} \in (158-k, 159-k], \\ \frac{9(1511-n)}{85} &= 159\frac{84}{85} - k - \frac{\beta}{85} \in [159-k, 160-k), \end{aligned}$$

and from

$$159 - k \in \left[\frac{2(1511-n)}{19}, \frac{9(1511-n)}{85}\right]$$

we see that the integer $1511-n$ is coloured red.

Therefore the point $A = 755\frac{1}{2}$ has the desired properties.

2.4. Consider the following two possibilities.

a) Let any two persons speak at least one common language. The person A speaks with the remaining 1984 people a common language, and speaks at most five languages, so at least there is one language spoken by at least

$$\left\lfloor \frac{1984}{5} \right\rfloor > 200$$

participants of the congress.

b) The second possibility is that there are two persons A and B who do not speak any language in common. Then every of the remaining 1983 persons can speak to at least one of the persons A and B (this follows from the three men condidition). So at least 992 participants can speak with one of these two persons (say A). This implies that A can speak the same language to at least 199 people, because otherwise he could speak to at most $5 \cdot 198 < 992$ people. These group together with the person A constitutes the desired 200 participants speaking the same language.

3.1. In order to prove the inequality, note that

$$\begin{aligned} DF &= IF - r \\ EG &= IG - r \\ IF \cdot IG &= R^2 - IO^2, \end{aligned}$$

Figure 3.1.

where R is a radius of the circle circumscribed to the triangle ABC (the last relation represents a well known property of the power of point I with respect to the circle). Then

$$DF \cdot EG \geq r^2$$

is equivalent to

$$FG \leq \frac{R^2 - IO^2}{r}.$$

As the line given in a problem is arbitrary, we have to prove that

$$2R \leq \frac{R^2 - IO^2}{r},$$

i.e.

$$IO^2 \leq R(R - 2r).$$

But, according to the well known Euler's formula, we have $IO^2 = R(R - 2r)$!

The equality holds true if and only if the straight line passes through O and I.

3.2. First describe a sphere around the tetrahedron $ABCD$. Its intersection with the plane ABC is a circle k_D circumscribed to the triangle ABC. Denote its center by O_D and radius by r_D.

The same for the remaining three triangles (sides of $ABCD$).

Figure 3.2.a)

Figure 3.2.b)

The condition

$$AE \cdot EB = BF \cdot FC = CG \cdot GA$$

means that powers of points E, F, G are the same with respect to k_D. If distances of E, F and G from O_D are e, f and g respectively, then we have

$$(r_D - e)(r_D + e) = (r_D - f)(r_D + f) = (r_D - g)(r_D + g).$$

From this we conclude $e = f = g$, so that points E, F and G lie on a circle of radius $\tilde{r}_D$, concentric to k_D. The same for the remaining three triangles.

Figure 3.2.c)

Let d_E, $d_F, \ldots$ be distances of points E, $F \ldots$ from the center O. Now as OO_D is perpendicular to the side ABC, we have

$$d_F^2 = \tilde{r}_D^2 + OO_D^2 = d_E^2 = \tilde{r}_C^2 + OO_C^2 = d_H^2.$$

The same holds for all the other pairs of points from the problem, which means that they lie on a sphere with a center at O.

3.3. It is clear that $a_{n-1} \neq 0$ implies $a_{n+1} \neq 0$, so that $a_n \neq 0$ for all n. Note first that for all n we have

$$\frac{a_n^2 + a_{n+1}^2 + c}{a_n a_{n+1}} = \frac{a_n^2 + (\frac{a_n^2+c}{a_{n-1}})^2 + c}{a_n \frac{a_n^2+c}{a_{n-1}}} = \frac{a_{n-1}^2 + a_n^2 + c}{a_{n-1}a_n}.$$

By induction we see that

$$\frac{a_n^2 + a_{n+1}^2 + c}{a_n a_{n+1}} = \frac{a^2 + b^2 + c}{ab}.$$

a) To prove the sufficiency of the condition in the problem, note that

$$\begin{aligned} a_{n+1} &= \frac{a_n^2 + c}{a_{n-1}} \\ &= \frac{a_{n-1}^2 + a_n^2 + c}{a_{n-1}a_n} a_n - a_{n-1} \\ &= \frac{a^2 + b^2 + c}{ab} a_n - a_{n-1}. \end{aligned}$$

By induction we get that all a_n are in $\mathbf{Z}$.

b) Conversly, assume that $a_n \in \mathbf{Z}$ for all n. Let

$$\frac{a^2 + b^2 + c}{ab} = \frac{p}{q},$$

where $p \in \mathbf{Z}$, $q \in \mathbf{N}$, are relatively prime. Then we have

(1) $$a_{n+1} = \frac{p}{q} a_n - a_{n-1}, \quad \forall n \geq 2.$$

From this we see that $\frac{p}{q} a_n \in \mathbf{Z}$, i.e. $a_n = q a_n^{(1)}$, where $a_n^{(1)} \in \mathbf{Z}$. From (1) we get

(2) $$a_{n+1}^{(1)} = \frac{p}{q} a_n^{(1)} - a_{n-1}^{(1)}, \quad \forall n \geq 3.$$

We have again that $a_n^{(1)} = q a_n^{(2)}$, i.e. $a_n = q^2 a_n^{(2)}$. Repeating this procedure, we arrive to

$$a_n = q^k a_n^{(k)}, \quad \forall n \geq k+1,$$

Let us write shorter $a_{k+1} = q^k b_{k+1}$. Now note that

$$c = a_{k+1} a_{k-1} - a_k^2 \in \mathbf{Z},$$

so that

$$\frac{p}{q}=\frac{a_k^2+a_{k+1}^2+c}{a_ka_{k+1}}=\frac{q^{2k}b_{k+1}^2+q^{2k-2}b_k^2+c}{q^{2k-1}b_kb_{k+1}}.$$

From this it immediately follows that q^{2k-2} must devide c for every k, and this is possible only for $q=1$.

3.4. a) It is clear that the triangle satisfying the conditions of the problem cannot be degenerated. Let P be the area of each of the triangles TAB, TBC, TCA (these areas are equal). The line AT divides the side BC into two parts of lengths a_1 and a_2, and the triangle BTC into two triangles of areas P_1 and P_2. Denote the altitudes of triangles ABA' and BTC, drawn from A and T respectively, by v and v'. Then from

Figure 3.4.a$_1$)

$$\frac{a_1v}{2}=P+P_1,\quad \frac{a_2v}{2}=P+P_2$$

we have

$$\frac{a_1}{a_2}=\frac{P+P_1}{P+P_2}. \tag{1}$$

Also from

$$\frac{a_1v'}{2}=P_1,\quad \frac{a_2v'}{2}=P_2$$

we get

$$\frac{a_1}{a_2}=\frac{P_1}{P_2}. \tag{2}$$

Figure 3.4.a$_2$)

Now relations (1) and (2) yield $P_1 = P_2$, i.e. $a_1 = a_2$, and in a similar way for the remaining sides of the triangle ABC. So T must be the barycenter of the triangle ABC.

From the condition in the problem we have

$$c + \frac{2}{3}(t_a + t_b) = a + \frac{2}{3}(t_b + t_c),$$

i.e.

$$c - a = \frac{2}{3}(t_c - t_a).$$

Assuming $c \neq a$, it is easy to see that $t_c \neq t_a$. From

$$(c-a)(t_c + t_a) = \frac{2}{3}(t_c^2 - t_a^2) = \frac{2}{3}\left(\frac{a^2+b^2}{2} - \frac{c^2}{4} - \frac{b^2+c^2}{2} + \frac{a^2}{4}\right) = \frac{1}{2}(a^2 - c^2)$$

we get

$$t_a + t_c = -\frac{a+c}{2},$$

which is impossible. So $c = a$ and similarly $a = b$. Therefore, the triangle ABC is isosceles.

b) It is clear that the point T cannot lie on the boundary of the triangle ABC. Otherwise we would not have equality of all the areas, because of the degeneration of some of the triangles.

Let us first show that T can be neither in sector I, nor on lines AB, AC, BC.

Figure 3.4.b$_1$)

Figure 3.4.b$_2$)

Indeed, assume by contradiction that T is in the sector I. Then the line TA intersects the side BC in point D. Without loss of generality we can assume that $D \neq C$. Now we have

$$P(TAC) = P(TBC) \geq P(TDC) = P(TAC) + P(ADC) > P(TAC),$$

where P is the corresponding area, and this is a contradiction.

Figure 3.4.b$_3$)

We can therefore take that the point T lies in the sector II, say opposite to B. Denote the intersection of lines BT and AC by D. Let x, y, z, u be lengths of sides and P_1, P_2, P_3, P_4 areas of triangles like on the picture. Then we have

$$P_1 + P_2 = P_3 + P_4 = P_2 + P_3,$$

from which it follows

$$P_1 = P_3, \quad P_2 = P_4.$$

This implies (using for instance the sine theorem) $xy = uz$, $xu = yz$. Multiplying these relations we get $x = z$, $y = u$, i.e. $ABCT$ is a parallelogram. But we also have

$$a + c + 2y = a + c + 2x$$

i.e. $x = y$. Therefore the point B lies on the circle of the diameter AC, so that ABC is a right–angle triangle and T is the fourth vertix of $ABCT$, having the desired property.

4.1. Substituting $y = 0$ and then $x = a$ into the relation, we have

$$f(x) = f(x)f(a) + f(0)f(a - x) \tag{1}$$

$$f(a) = [f(a)]^2 + \frac{1}{4}.$$

Therefore $(f(a) - 1/2)^2 = 0$ and $f(a) = 1/2$. From (1) we obtain

$$f(x) = f(a - x). \tag{2}$$

So the relation given in the problem now becomes

$$f(x + y) = 2f(x)f(y).$$

Now we have

$$\frac{1}{2} = f(a) = 2f(x)f(a-x) = 2[f(x)]^2,$$

and from this $f(x) = \pm 1/2$.

If we had b such that $f(b) = -1/2$, then we would have

$$-\frac{1}{2} = f(b) = f(\frac{b}{2} + \frac{b}{2}) = 2f(\frac{b}{2})^2,$$

which is impossible. Therefore $f(x) = 1/2$ for all x.

Alternate solution. Substituting $x = y = 0$ we obtain $f(a) = \frac{1}{2}$. Now change y by 0 and next by a in the relation. Thus we get

$$f(x) = f(a-x), \quad f(x) = f(a+x)$$

and for any real x it follows

$$f(-x) = f(a-(-x)) = f(a+x) = f(x).$$

Let x and y be arbitrary real numbers. Using the above together with the identity in the problem, we have

$$\begin{aligned} f(x-y) &= f(x)f(a+y) + f(-y)f(a-x) \\ &= f(x)f(a-y) + f(y)f(a-x) = f(x+y) \end{aligned}$$

Putting $y = x$ into this relation we finally obtain $f(2x) = f(0) = 1/2$, i.e. $f(x) = 1/2$ for all x.

4.2. First we have

$$b - a = \frac{2}{\sqrt{x+1} + \sqrt{x-1}} + \frac{2}{\sqrt{y+1} + \sqrt{y-1}} \geq 2.$$

Without loss of generality we can assume that $x \geq y$. Therefore

$$\frac{2}{\sqrt{y+1} + \sqrt{y-1}} \geq 1.$$

After squaring twice the inequality $2 - \sqrt{y-1} \geq \sqrt{y+1}$, we obtain $y \leq 5/4$.

If we had $b - a \geq 3$, then we would have

$$\frac{2}{\sqrt{y+1} + \sqrt{y-1}} \geq \frac{3}{2},$$

i.e. $4/3 - \sqrt{y-1} \geq \sqrt{y+1}$, and after squaring

$$\frac{8}{3}\sqrt{y-1} + \frac{2}{9} \leq 0,$$

which is impossible.

So we have $b - a = 2$. Denote

$$\begin{aligned} x + y &= s \\ xy &= p. \end{aligned}$$

Then

$$\begin{aligned} a^2 &= x - 1 + y - 1 + 2\sqrt{xy - x - y + 1} \\ b^2 &= a^2 + 4a + 4 = x + 1 + y + 1 + 2\sqrt{xy + x + y + 1}. \end{aligned}$$

i.e.

$$\begin{aligned} a^2 + 2 - s &= 2\sqrt{p - s + 1} \geq 0 \\ a^2 + 4a + 2 - s &= 2\sqrt{p + s + 1} \end{aligned}$$

and from this after squaring and substracting both relations we obtain

$$a^3 + 2a^2 + 2a = (a+1)s \leq (a+1)(a^2+2).$$

Finally, we obtain $a^2 \leq 1$, that is $a = 1$, and from this $s = 5/2$, $p = 25/16$, $x = y = 5/4$.

REMARK We can obtain $b = a + 2$ more directly by noting that the obvious inequality

$$\sqrt{t+1} + \sqrt{t-1} \geq \sqrt{2}, \quad \forall t \geq 1$$

implies

$$0 < b - a = \frac{2}{\sqrt{x+1} + \sqrt{x-1}} + \frac{2}{\sqrt{y+1} + \sqrt{y-1}} \leq 2\sqrt{2}.$$

4.3. For every s and t such that $0 < s < t < \pi/2$, we have $\sin^{23} s < \sin^{23} t$ and $\cos^{48} s > \cos^{48} t$. Therefore the function

$$f(t) = \frac{\sin^{23} t}{\cos^{48} t}$$

is strictly increasing on $(0, \pi/2)$. So from $f(\alpha/2) = f(\beta/2)$ we get $\alpha = \beta$, that is $AC/BC = 1$.

Figure 4.4.

4.4. Let M be the midpoint of AC and $AM = x$, $\angle AMO_1 = \varphi$. As the angle $\angle AMO_2$ is equal to $\pi/2$, we have $MO_2 = \sqrt{2 - x^2}$. From the cosine theorem for triangle $\triangle O_1O_2M$ we get:

$$\sin\varphi = -\cos(\frac{\pi}{2} + \varphi) = \frac{1 + x^2}{2\sqrt{2 - x^2}}.$$

As the triangle AMO_1 is isosceles, we also have $\cos\varphi = \frac{x}{2}$, i.e. $\sin\varphi = \sqrt{1 - x^2/4}$ (note that $0 < \varphi < \pi/2$). Introduce a new variable $t = x^2$, $0 < t < 2$. The unique solution of

$$\frac{1 + t}{2\sqrt{2 - t}} = \sqrt{1 - \frac{t}{4}}$$

is $t = 7/8$. Therefore $x = \sqrt{t} = \sqrt{7/8}$ and

$$AC = 2x = \sqrt{\frac{7}{2}}.$$

5.1. Let $AB = c$, $AC = b$ and $BC = a$. Without loss of generality we may assume that $0 < b < a$, so that the points H and L lie on the segment AM. From the conditions of the problem we have

Figure 5.1.

$$\frac{P(HMC)}{P(ABC)} = \frac{HM \cdot HC}{AB \cdot HC} = \frac{HM}{AB} = \frac{1}{4}$$

$$\frac{P(LMC)}{P(ABC)} = \frac{LM \cdot HC}{AB \cdot HC} = \frac{LM}{AB} = 1 - \frac{\sqrt{3}}{2}.$$

It is well known that for bisectors we have the following relation:

$$\frac{AL}{b} = \frac{c - AL}{a},$$

so that

$$AL = \frac{bc}{a+b}.$$

Therefore

$$LM = \frac{c}{2} - AL = \frac{c(a-b)}{2(a+b)}.$$

As $LM/c = 1 - \sqrt{3}/2$, we obtain that $a = \sqrt{3}\,b$. On the other hand, from the cosine theorem

$$a^2 = b^2 + c^2 - 2bc\cos\alpha = b^2 + c^2 - 2c \cdot AH$$

follows

$$HM = \frac{c}{2} - AH = \frac{a^2 - b^2}{2c} = \frac{b^2}{c},$$

and from this

$$\frac{1}{4} = \frac{HM}{c} = \frac{b^2}{c^2},$$

i.e. $c = 2b$. Therefore $\alpha = 60\deg$, $\beta = 30\deg$, $\gamma = 90\deg$.

Alternate solution.

1. From $AH = HM$ we obtain $\angle HMC = \alpha$, and from this $CM = AC = b$.

2. As $LM = (1 - \sqrt{3}/2)c$, we have $LB = \frac{3-\sqrt{3}}{2}c$ and from $AL : LB = b : a$ we deduce

$$a = \sqrt{3}\,b.$$

3. Now $HB = HM + MB = 3HM = 3AC\cos\alpha$ and $HB^2 + HC^2 = BC^2$, so that $9\cos^2\alpha + \sin^2\alpha = 3$, i.e. $\cos\alpha = 1/2$ ($\cos\alpha$ cannot be negative). Hence $\alpha = 60\deg$.

4. We conclude that $AM = MC = MB$, and therefore M is the center of the circumscribed circle, i.e. $\gamma = 90\deg$.

5.2. First we note that

$$P(tx, ty) = P(t, 0)P(x, y) \tag{1}$$

$$P(ts, 0) = P(t, 0)P(s, 0) \tag{2}$$

Define $p(x) = P(x, 0) = a_k x^k + \cdots + a_0$. From (2) one gets easily (comparing coefficients on both sides) that either $p(x) = 0$, or $p(x) = 1$, or $p(x) = x^k$ with $k \geq 1$, for all x.

Therefore, either $P(x, y) = 0$, or $P(x, y) = 1$, or

$$P(tx, ty) = t^k P(x, y) \tag{3}$$

Consider (3):

$$P(x, y)P(1, 1) = P(x + y, x + y) = (x + y)^k P(1, 1)$$
$$P(x, y)P(1, -1) = P(x - y, x - y) = (x - y)^k P(1, -1).$$

Now we have three cases:

a) If $P(1, 1) \neq 0$, then $P(x, y) = (x + y)^k$, which is easily verified to satisfy the condition of the problem.

b) If $P(1, -1) \neq 0$, then $P(x, y) = (x - y)^k$, which also satisfyes the condition in the problem.

c) Assume that $P(1, 1) = P(1, -1) = 0$. Keeping y fixed, and dividing $P(x, y)$ by $x^2 - y^2$ we obtain

$$P(x, y) = (x^2 - y^2)Q(x, y) + xR(y) + S(y),$$

for all $x, y \in \mathbf{R}$. Substituting $(x,y) = (t,t)$ and $(x,y) = (t,-t)$ and using (3) we obtain

$$\begin{aligned} tR(t) + S(t) &= 0 \\ -tR(t) + S(t) &= 0, \end{aligned}$$

from which we obtain $R(t) = S(t) = 0$ for all t. Therefore

$$P(x,y) = (x^2 - y^2)Q(x,y)$$

in this case.

The polynomial Q also satisfies the condition in the problem like P, and its degree is equal to $k-2$. So, we can again repeat the whole procedure with three possible cases for Q, instead of P. After finitely many steps we arrive to

$$P(x,y) = (x+y)^m (x-y)^n, \tag{4}$$

where $m, n \geq 0$ are integers. Therefore all solutions of the problem are constants 0, 1 and polynomials given by (4).

REMARK Let us state the problem analogous to the preceding one: Find all polynomials $P(x,y,z)$ in three variables, such that

$$P(a,b,c) \cdot P(x,y,z) = P(ax+bz+cy, ay+bx+cz, az+by+cx)$$

for all real numbers a, b, c, x, y, z. It can be shown that all solutions are either constants 0 and 1, or

$$P(x,y,z) = (x+y+z)^m (x^2+y^2+z^2-xy-yz-zx)^n,$$

where $m, n \geq 0$ are integers. Try to prove this!

5.3. Denote $a_1 = A_1A_2$, $a_2 = A_3A_4$, $a_3 = A_1A_4$, $a_4 = A_2A_3$, $a_5 = A_2A_4$, $a_6 = A_1A_3$.

Let M_1, M_2, M_3, M_4 be midpoints of edges indicated on the diagram.These are the vertices of a parallelogram.

Analogously, we have two more midpoints M_5 and M_6 and the corresponding parallelograms.

The opposite edges of the tetrahedron define nonparallel lines. Denote the smallest distance of these three pairs of nonparallel lines by d. It is clear that the pair of lines with minimal distance defines two parallel planes containing them and the whole tetrahedron is in between. Therefore

$$\begin{aligned} d &\leq M_1M_2 \\ d &\leq M_3M_4 \\ d &\leq M_5M_6. \end{aligned}$$

Figure 5.3.

Using the parallelogram equality we obtain

$$\begin{aligned}3d^2 &\le |M_1M_2|^2 + |M_3M_4|^2 + |M_5M_6|^2\\ &= \frac{1}{2}(|M_1M_2|^2 + |M_3M_4|^2) + \frac{1}{2}(|M_1M_2|^2 + |M_5M_6|^2)\\ &\quad + \frac{1}{2}(|M_3M_4|^2 + |M_5M_6|^2)\\ &= \frac{1}{4}[(a_5^2 + a_6^2) + (a_3^2 + a_4^2) + (a_1^2 + a_2^2)] = \frac{P}{4},\end{aligned}$$

i.e. $d \le \frac{1}{2}\sqrt{P/3}$.

Alternate solution. The opposite edges of the tethraedhron define two parallel planes containg them. These six plains define a parallelepiped. Denote lengths of its edges by a, b, c, and let d be the smallest of distances between three pairs of parallel planes introduced above. Then by the parallelogram identity the sum of squares of opposite edges is equal to

$$2a^2 + 2c^2, \quad 2a^2 + 2b^2, \quad 2b^2 + 2c^2.$$

From this we have

$$P = 4(a^2 + b^2 + c^2) \geq 4(d^2 + d^2 + d^2) = 12d^2.$$

5.4. We shall prove that the single pair of consecutive terms which verifies the given condition is (a_7, a_8).

Let us put $q - p = s - r = x$. Then we have

$$a_n = p(p + x), \quad a_{n+1} = r(r + x).$$

The function $p \mapsto p(p+x)$ is increasing, therefore $p < r$. For every odd $n = 2k+1$ we have

$$a_n = 2^n + 49 \equiv 2^n + 1 = 2(3 + 1)^k + 1 \equiv 0 \pmod 3$$

and $2^n + 49$ is obviously odd. So, $p = 3$.

We have

$$a_{n+1} = 2a_n - 49 < 2a_n, \tag{1}$$

which implies $r < 2p$. Indeed, if $r \geq 2p$, then

$$a_{n+1} = r(r + x) \geq 2p(2p + x) > 2p(p + x) = 2a_n.$$

Hence $3 < r < 6$ and therefore $r = 5$. Substituting the values $p = 3$ and $r = 6$ into the recurrence relation (1), one finds $5(5 + x) = 6(3 + x) - 49$, hence $x = 56$. It follows that $a_n = 3 \cdot 59$ and $a_{n+1} = 5 \cdot 61$. One verifies directly that these numbers are $a_7 = 2^7 + 49$ and $a_8 = 2^8 + 49$.

Alternate solution. Denote $y = s - q = r - p$. Like in the previous solution we get

$$a_n = 3q, \quad a_{n+1} = (3 + y)(q + y).$$

Therefore $(3 + y)(q + y) = 2^{n+1} + 49 = 2^n + 3q$, i.e.

$$y\left(\frac{2^n + 49}{3} + y + 3\right) = 2^n. \tag{1}$$

If $y \geq 3$, then the left hand side of (1) is bigger than the right hand side, which is impossible. Also in case $y = 1$ we have a contradiction: $a_{n+1} = 4(q + 1)$. So the only possibility left is $y = 2$. Now from (1) we compute $n = 7$ and the result follows easily.

6.1. If $d_2 > 2$, then n, d_2, d_3 and d_4 are odd. So $d_1^2 + d_2^2 + d_3^2 + d_4^2$ is even, which is a contradiction. Therefore $d_2 = 2$, n is even and from the equality $n = 1^2 + 2^2 + d_3^2 + d_4^2$ we obtain that exactly one of the numbers d_3 and d_4 is even.

a) Let d_3 be even, i.e. $d_3 = 2a$, $a \geq 1$. Then $a < d_3$ is a divisor of n and hence $a = d_1 = 1$, or $a = d_2 = 2$. It is easy to check that both cases are impossible.

b) If d_4 is even, $d_4 = 2a$, $a \geq 1$, a similar reasoning shows that $a = 1$, $a = 2$ or $a = d_3$. The case $a = 1$ leads to a contradicition. If $a = 2$, then $d_4 = 4$, so that $d_3 = 3$. Therefore $n = 1^2 + 2^2 + 3^2 + 4^2 = 30$, which is not divisible by four.

Now it suffices to consider the case $a = d_3$. We have $d_4 = 2d_3$, and then

$$n = 1^2 + 2^2 + d_3^2 + (2d_3)^2 = 5(d_3^2 + 1).$$

Since d_3 divides n, we obtain that $d_3 = 5$, $d_4 = 10$, $n = 1^2 + 2^2 + 5^2 + 10^2 = 130$. All the divisors of 130 are $1, 2, 5, 10, 13, 26, 65, 130$. Hence, $n = 130$ is the unique solution.

6.2. Observe that if $P(x) = 0$, then $|x| < 9$. Indeed, if $|x| \geq 9$, then:

$$\begin{aligned}|P(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0| \\ &\geq 2|x|^n - 9(|x|^{n-1} + \cdots + 1) = \frac{2|x|^{n+1} - 11|x|^n + 9}{|x| - 1} \\ &> \frac{2|x|^n(|x| - 9) + 9}{|x| - 1} > 0.\end{aligned}$$

Suppose by contradiction that $P(x) = Q(x)R(x)$, where Q and R are polynomials with integer coefficients. We have

$$\overline{a_n a_{n-1} \dots a_0} = P(10) = Q(10)R(10).$$

By the above we have that $|Q(10)| > 1$ and $|R(10)| > 1$, which is impossible ($P(10)$ is prime). Namely, since $Q(x) = a\prod_{i<k}(x - x_i)$, $a \in \mathbf{Z}$, then $|x_i| < 9$ and so $|Q(10)| = |a|\prod_{i<k}|10 - x_i| > 1$.

REMARK The conclusion of the problem is valid also for the case $a_n = 1$, but it requires a different proof (see the problem book of Pólya–Szegö).

6.3. Let us denote the corresponding areas shortly in round brackets. Define

$$E_1 = (AB_1G), \quad E_2 = (AC_1G).$$

As G is the barycentre of ABC, the altitude of the triangle AGC_1 from the vertex G is one third of the altitude of triangle ABC_1 drawn from B. Similarly for triangles AB_1C and AB_1G. Therefore

$$(ABC_1) = 3E_2, \quad (AB_1C) = 3E_1. \tag{1}$$

Note that

$$\begin{aligned}(ABC_1) &= E_1 + E_2 + (BB_1GC_1) = 3E_2 \\ (AB_1C) &= E_1 + E_2 + (CC_1GB_1) = 3E_1,\end{aligned}$$

Figure 6.3.

so that

(2) $$(BB_1GC_1) + (CC_1GB_1) = E_1 + E_2.$$

Moreover

(3) $$\frac{(ABC)}{(ABC_1)} = \frac{AC}{AC_1}, \quad \frac{(ABC)}{(AB_1C)} = \frac{AB}{AB_1}.$$

From (1) and (3) we get

$$\frac{(ABC)}{E_2} = 3\frac{AC}{AC_1}, \quad \frac{(ABC)}{E_1} = 3\frac{AB}{AB_1}.$$

From this we have

$$\begin{aligned}\frac{E_1 + E_2}{(ABC)} &= \frac{1}{3}\left(\frac{AC_1}{AC} + \frac{AB_1}{AB}\right)\\ &\geq \frac{2}{3}\sqrt{\frac{AC_1 \cdot AB_1}{AC \cdot AB}}\\ &= \frac{2}{3}\sqrt{\frac{(AB_1C_1)}{(ABC)}}\\ &\geq \frac{2}{3}\sqrt{\frac{E_1 + E_2}{(ABC)}},\end{aligned}$$

and the desired inequality follows from (2). The equality holds if and only if $AC_1/AC = AB_1/AB$ and $(AB_1C_1) = E_1 + E_2$, that is if and only if the line ℓ is parallel to BC and contains G.

6.4. For the upper bound, let $\mathcal{F}$ be an arbitrary family of 3–element subsets of $\{1, \dots, n\}$, such that $|A \cap B| \leq 1$ for any $A, B \in \mathcal{F}$. All 2–element subsets of all elements of $\mathcal{F}$ must be different. Each $A \in \mathcal{F}$ contains $\binom{3}{2} = 3$ different 2–element subsets and all of these are required to be distinct. Therefore, since the total number of 2–element subsets of $\{1, \dots, n\}$ is $\binom{n}{2}$, we have

$$3|\mathcal{F}| \leq \binom{n}{2}. \tag{1}$$

As $\mathcal{F}$ was arbitrary, it follows that

$$f(n) \leq \frac{1}{3}\binom{n}{2} = \frac{n^2 - n}{6}, \tag{2}$$

which proves the upper bound.

For the lower bound, consider the family $\mathcal{F}_0$ of all 3–element subsets $A = \{a, b, c\}$ of $\{1, \dots, n\}$, such that $a + b + c = n$, or $a + b + c = 2n$. Evidently, if $a + b + c_1 \in \{n, 2n\}$ and $a + b + c_2 \in \{n, 2n\}$, then $c_1 = c_2$ (otherwise we would have $|c_1 - c_2| = n$, which contradicts the fact that $c_1, c_2 \in \{1, \dots, n\}$). Thus the family $\mathcal{F}_0$ has the desired property, and hence $f(n) \geq |\mathcal{F}_0|$. Now we estimate $|\mathcal{F}_0|$.

For selecting an arbitrary $A = \{a, b, c\} \in \mathcal{F}_0$, we have n possibilities for a, and (for fixed a) at least $n - 4$ possibilities for b (since $b \neq a$, $b \neq (n - a)/2$, $b \neq (2n - s)/2$, $b \neq s - 2a$, and $1 \leq b \leq n$, where $s = a + b + c \in \{n, 2n\}$; these conditions come from the fact that $\{a, b, c\}$ must be a 3–element subset, that is $a \neq b$, $b \neq c$ and $c \neq a$). But in this way we select ordered triples (a, b, c) in $\mathcal{F}_0$. Since a 3–element subset can be permuted in $3! = 6$ ways, it follows that any $A \in \mathcal{F}_0$ is counted exactly 6 times. By the above we obtain

$$6|\mathcal{F}_0| \geq n(n - 4),$$

that is

$$f(n) \geq |\mathcal{F}_0| \geq \frac{n^2 - 4n}{6}.$$

This proves the lower bound and concludes the solution of the problem.

REMARK 1. We can obtain slightly better lower bound as follows. Observe that the condition $b \neq (n - a)/2$ in the argument above is “effective” only for those a, $1 \leq a \leq n$, with $a \equiv n \pmod 2$. Also, the condition $b \neq (2n - a)/2$ is “effectove” only for even a. It follows that for odd n (and fixed a) b can be selected in at least

$$\begin{cases} n - 2 \text{ ways, for odd } a \\ n - 4 \text{ ways, for even } a. \end{cases}$$

Using a similar argument as in the solution above, we obtain thus

$$f(n) = \begin{cases} \frac{n(n-3)}{6} & \text{for odd } n \\ \frac{1}{6}\left[\frac{n}{2}(n-2) + \frac{n}{2}(n-4)\right] = \frac{n(n-3)}{6} & \text{for even } n. \end{cases}$$

Hence, finally

$$f(n) \geq \frac{n(n-3)}{6}.$$

Alternate solution and generalization. Let us prove the lower bound defining again

$$\mathcal{F}_0 = \left\{ \{a,b,c\} \subseteq \{1,2,\ldots,n\} \colon a+b+c \in \{n,2n\},\ a \neq b \neq c \neq a \right\}.$$

As we saw in previous solution, the set $\mathcal{F}_0$ has properties (i) and (ii). The idea is to compute effectively the cardinality of this set.

It is clear that any $\{a,b,c\} \in \mathcal{F}_0$ defines precisely six different ordered triples of its elements. Let us count all such ordered triples (a,b,c) from $\mathcal{F}_0$.

If $a, b \in \{1,\ldots,n\}$ are any two elements for which $a \neq b$, then c is uniquely determined:

1) if $a+b < n$, then $c = n - (a+b)$;
2) if $a+b \geq n$, then $c = 2n - (a+b)$.

From all such pairs (a,b) we shall have to exclude those, for which the corresponding c is equal to a or b (see the definition of $\mathcal{F}_0$). We are led to the following two cases:

1') Let $a+b<n$. Then $c=a$ iff $2a+b=n$ and $c=b$ iff $a+2b=n$;
2') Let $a+b \geq n$. Then $c=a$ iff $2a+b=2n$ and $c=b$ iff $a+2b=2n$.

So, the desired number of ordered triples (a,b,c) will be equal to the number of pairs which are left in the 'square'

$$\left\{ (a,b) \colon a,b \in \{1,\ldots,n\} \right\}$$

when we remove points on 'lines'

$$\begin{gathered} a = b \\ 2a+b=n, \quad a+2b=n, \\ 2a+b=2n, \quad a+2b=2n. \end{gathered}$$

Let us count all such pairs (a,b). We consider the following four cases.

I. $n = 2k \not\equiv 0 \pmod 3$. From the 'square' having $n^2 = (2k)^2$ points we have to remove

- $2k$ points on 'line' $a=b$;
- $k-1$ points on 'line' $a+2b=n$ (for $b=1,\ldots,k-1$) and $k-1$ points on 'line' $2a+b=n$ (for $a=1,\ldots,k-1$);

- k points on 'line' $a + 2b = 2n$ (for $b = k, \dots, 2k-1$) and k points on 'line' $2a + b = 2n$ (for $a = k, \dots, 2k-1$).

We recommend you to draw the 'square' and 'lines' for, say, $n = 8$.

It is left to see that among points removed there are no those counted twice. Indeed, if we had

$$2a + b = a + 2b = n,$$

then we would have $a = b = n/3$, a contradiction. Therefore

$$6|\mathcal{F}_0| = (2k)^2 - 2k - 2(k-1) - 2k = n^2 - 3n + 2.$$

The remaining cases can be treated in a similar way, so we leave the details to the reader.

II. $n = 2k + 1 \not\equiv 0 \pmod 3$. By repeating the procedure like in I. we arrive to

$$6|\mathcal{F}_0| = (2k+1)^2 - (2k+1) - 4k = n^2 - 3n + 2.$$

III. $n = 2k \equiv 0 \pmod 3$. In this case we have

$$6|\mathcal{F}_0| = (2k)^2 - (2k) - 2(k-2) - 2(k-1) = n^2 - 3n + 6.$$

IV. $n = 2k + 1 \equiv 0 \pmod 3$. Here we have

$$6|\mathcal{F}_0| = (2k+1)^2 - (2k+1) - 4(k-1) = n^2 - 3n + 6.$$

Finally, we arrive to the following conclusion

$$|\mathcal{F}_0| = \begin{cases} \frac{n^2-3n+2}{6}, & n \not\equiv 0 \pmod 3 \\ \frac{n^2-3n+6}{6}, & n \equiv 0 \pmod 3, \end{cases}$$

i.e.

$$|\mathcal{F}_0| = \left\lfloor \frac{n^2 - 3n}{6} \right\rfloor + 1.$$

From this it is clear that

$$f(n) > \frac{n^2 - 3n}{6} \geq \frac{n^2 - 4n}{6}.$$

REMARK 2 It should be noted that the family $\mathcal{F}_0$ is not necessarily maximal, i.e. its cardinality could be less than $f(n)$. For instance, if $n = 8$, then by the above formula $|\mathcal{F}_0| = 7$. On the other hand, the family

$$\mathcal{F} = \big\{ \{1,2,8\},\ \{2,3,4\},\ \{4,5,6\},\ \{6,7,8\},\\ \{1,3,6\},\ \{2,5,7\},\ \{1,4,7\},\ \{3,5,8\} \big\}$$

satisfies the conditions (i) and (ii), and contains 8 elements.

7.1. By direct calculation we get

$$\begin{aligned} a_1 &\equiv 1 \pmod{11}\\ a_2 &\equiv 3 \pmod{11}\\ a_3 &\equiv 9 \pmod{11}\\ a_4 &\equiv 0 \pmod{11}\\ a_5 &\equiv 10 \pmod{11}\\ a_6 &\equiv 4 \pmod{11}\\ a_7 &\equiv 6 \pmod{11}\\ a_8 &\equiv 0 \pmod{11}\\ a_9 &\equiv 1 \pmod{11}\\ a_{10} &\equiv 0 \pmod{11}\\ a_{11} &\equiv 0 \pmod{11}. \end{aligned}$$

So from the relation in the problem we obtain that for every $n \geq 10$ we have $a_{n+2} \equiv 0 \pmod{11}$. The answer is:

$$n \in \{4, 8\} \cup \{n \in \mathbf{N} \colon n \geq 10\}.$$

Alternate solution. From the recurrence relation in the problem we immediately see that for $n \geq 3$ we have

$$\begin{aligned} a_n - a_{n-1} &= n(a_{n-1} - a_{n-2})\\ a_{n-1} - a_{n-2} &= (n-1)(a_{n-2} - a_{n-3})\\ &\vdots\\ a_4 - a_3 &= 4(a_3 - a_2)\\ a_3 - a_2 &= (a_2 - a_1) \end{aligned}$$

After consecutive substitutions we arrive to:

$$a_n - a_{n-1} = 3 \cdot 4 \ldots \cdot n \cdot (a_2 - a_1) = n!$$

i.e.

$$a_n = a_{n-1} + n!, \tag{1}$$

and therefore

$$a_n = 1! + 2! + 3! + \cdots + n!$$

Now we have

$$a_1 = 1 \quad a_2 = 3 \quad a_3 = 9 \equiv -2$$
$$a_4 \equiv -2 + 2 \cdot 3 \cdot 4 = 0 \quad a_5 \equiv 2 \cdot 3 \cdot 4 \cdot 5 \equiv -1 \quad a_6 \equiv -1(-1) \cdot 6 \equiv -7 \equiv 4$$

and in a similar way $a_7 \equiv 6$, $a_8 \equiv 0$, $a_9 \equiv 1$, $a_{10} \equiv 0$. All congruences are modulo 11. As for $n \geq 11$ we have $n! \equiv 0$, from (1) it follows that then $a_n \equiv 0$, as well as for $n = 4, 8$.

7.2. Defining $b_i = ix^i$ we can write:

$$(x + 2x^2 + \cdots + nx^n)^2 = \sum_{i=1}^{n} b_i \cdot \sum_{n=1}^{n} b_i = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j.$$

The power of the monomial $b_i b_j$ is equal to $i+j$. Hence, instead of summing up numbers a_i, $i = n+1, \ldots, 2n$, it suffices to sum up the products $b_i b_j$, for which $x = 1$ and $i + j \geq n + 1$. As $b_i b_j\,|_{x=1} = ij$, we can write

$$\begin{aligned}
\sum_{i=n+1}^{2n} a_i = \sum_{\substack{i,j\leq n\\ i+j>n}} ij = \sum_{i=1}^{n}\sum_{j=n-i+1}^{n} ij = \\
&= \sum_{i=1}^{n} i \cdot \frac{i(2n-i+1)}{2} = \frac{2n+1}{2}\sum_{i=1}^{n} i^2 - \frac{1}{2}\sum_{i=1}^{n} i^3 \\
&= \frac{2n+1}{2} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{1}{2}\frac{n^2(n+1)^2}{4} \\
&= \frac{1}{24} n(n+1)(5n^2+5n+2),
\end{aligned}$$

where we used well known summation identities, that are easy to prove by induction.

Alternate solution. Let us prove the claim by induction. For $n = 1$ it is checked directly. Denote the polynomial introduced in the problem by $f_n(x)$ and define

$$S_n = a_{n+1} + \cdots + a_{2n}.$$

Suppose that the claim is valid for n. Then we have

$$\begin{aligned}
f_{n+1}(x) &= b_0 + b_1 x + \cdots + b_{2n+2}x^{2n+2} = \\
&= (x + \cdots + nx^n + (n+1)x^{n+1})^2 = \\
&= (x + \cdots + nx^n)^2 + (n+1)^2 x^{2n+2} + 2(x + \cdots + nx^n)(n+1)x^{n+1} \\
&= a_0 + a_1 x + \cdots + a_{2n}x^{2n} \\
&\quad + [2(n+1)x^{n+2} + \ldots 2nx^{2n+1} + (n+1)^2 x^{2n+2}],
\end{aligned}$$

Figure 7.3.

from which it follows

$$
(1) \qquad \begin{aligned} S_{n+1} &= b_{n+2} + \cdots + b_{2n+2} \\ &= a_{n+2} + \cdots + a_{2n} + 2(n+1)(1 + \cdots + n) + (n+1)^2 \\ &= (S_n - a_{n+1}) + (n+1)^3. \end{aligned}
$$

It is only left to determine a_{n+1}, i.e. the coefficient at x^{n+1} for the polynomial

$$f_n(x) = (x + 2x^2 + \cdots + nx^n) \cdot (x + 2x^2 + \cdots + nx^n).$$

Substituting

$$
\begin{aligned} a_{n+1} &= 1 \cdot n + 2 \cdot (n-2) \cdots + n \cdot 1 = \sum_{i=1}^{n} i(n+1-i) = \\ &= (n+1)\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 = (n+1)\frac{n(n+1)}{2} - \frac{1}{6}n(n+1)(2n+1) \\ &= \frac{1}{6}n(n+1)(n+2) \end{aligned}
$$

into (1), we obtain

$$S_{n+1} = \frac{1}{24}(n+1)(n+2)[5(n+1)^2 + 5(n+1) + 2],$$

which proves the statement.

7.3. a) Let us first prove that the altitudes of the triangle $\triangle ABC$ bisect the angles $\angle C_1A_1B_1$, $\angle A_1B_1C_1$ andi $\angle B_1C_1A_1$. Let H be the orthocenter of $\triangle ABC$.

It suffices to prove that $\angle B_1C_1C = \angle A_1C_1C$. As $\angle HC_1B = \angle HA_1B = \pi/2$, the quadrelateral HC_1BA_1 is inscribed, so that $\angle HC_1A_1 = \angle HBA_1 = \angle B_1BA_1$. Similarly $\angle HC_1A = \angle HB_1A$, which implies that the quadrelateral HB_1AC_1 is also inscribed, and then $\angle B_1C_1H = \angle B_1AH = \angle B_1AA_1$. Finally $\angle BB_1A = \angle AA_1B = \pi/2$, so that as ABA_1B_1 is inscribed, we conclude

$$\angle B_1C_1C = \angle B_1C_1H = \angle B_1AA_1 = \angle B_1BA_1 = \angle HC_1A_1 = \angle A_1C_1C,$$

which proves the claim a).

b) Denote by S the center of the incircle. From a) we have that H is the center of the circle inscribed to the triangle $\triangle A_1B_1C_1$, and it is at the same time circumscribed to the triangle $\triangle A_2B_2C_2$. The sides B_1C_1 and A_1C_1 are tangent to this circle, so that $HA_2 \perp B_1C_1$ and $HB_2 \perp A_1C_1$. As HC_1 is a bisector of the angle $\angle A_2C_1B_2$, the triangles $\triangle HA_2C_1$ and $\triangle HB_2C_1$ are similar. Therefore HC_1 is a 'line' of symmetry of A_2B_2, so that we have $A_2B_2 \perp HC_1 \perp AB$ and then $A_2B_2 \| AB$. Similarly $B_2C_2 \| BC$ and $C_2A_2 \| CA$, so that the triangles $\triangle ABC$ and $\triangle A_2B_2C_2$ have parallel sides. But then the Euler's "line's' are parallel, and as H is the orthocenter of $\triangle ABC$ and the center of the circle inscribed to $\triangle A_2B_2C_2$, these two "line's' coincide.

7.4. For integers 1, 3, 6, 8, differences of any two are primes. According to the condition in our problem, $f(1)$, $f(3)$, $f(6)$, $f(8)$ must all be mutualy different, so that A has at least four elements.

Let us define $A = \{1, 2, 3, 4\}$ and $f(n) \equiv n \pmod 4$. From $f(i) = f(j)$ we conclude $i \equiv j \pmod 4$, i.e. four divides $|i-j|$, so that $|i-j|$ not prime. Therefore the minimal cardinality of A is 4.

8.1. Let us prove that $\overline{MK} = \overline{KL} = \overline{MN} = \overline{NP}$. From

$$\angle OAB = \angle OBA = \frac{1}{2}(\pi - 2\angle ACB) = \frac{\pi}{2} - \angle ACB,$$
$$\angle AKL = \angle BNP = \angle ACB$$

we obtain

$$\overline{MK} = \overline{MN} = \overline{KL}. \tag{1}$$

The triangles AKL and PNB are congruent (they are both conguent to ABC, so that

$$\overline{AK} \cdot \overline{BN} = \overline{KL} \cdot \overline{PN}. \tag{2}$$

The triangles AMK and MBN are congruent as well, therefore

$$\overline{AK} \cdot \overline{BN} = \overline{MK} \cdot \overline{MN}. \tag{3}$$

From (1), (2) and (3) we have $\overline{MK} = \overline{MN} = \overline{KL} = \overline{NP}$. Therefore $KN \parallel PL$, and so $\angle MLP = \angle MKN = \angle ACB$.

8.2. Let P be the area of the triangle and $s = \frac{1}{2}(a+b+c)$. By Heron's formula we have $P = \sqrt{s(s-a)(s-b)(s-c)}$, $(s-a)+(s-b)+(s-c) = s$. For any natural number k define $s-a = k^4$, $s-b = 4k^2$ i $s-c = 4$. We have $s = (k^2+1)^2$ and

$$a = 4(k^2+1), \quad b = k^4+4, \quad c = k^2(k^2+4).$$

Let k be odd and $k > 1$. Then a and c are relatively prime, which proves (i) and (ii). From $h_a = 2P/a$ and similarly for remaining altitudes, we get

$$h_a = \frac{2k^3(k^2+2)}{k^2+1}, \quad h_b = \frac{8k^3(k^2+2)}{k^4+4}, \quad h_c = \frac{8k(k^2+2)}{k^2+4}.$$

As $k^4+4 = (k^2-2)(k^2+2)+8$ and k is odd, fractions by which h_b and h_c are expressed cannot be canceled, while the fraction representing h_a can be canceled by 2. This proves (iii).

8.3. By convexity, the convex polygon in the probelm is contained inside the starshaped region on the picture, whose vertices are A_i, M_i, $i = 1, \ldots, 6$. The convexity implies also that the area of the part of convex polygon inside the union of congruent triangles $A_1M_1A_2$ i $A_2M_2A_3$, is not greater than any of its areas (prove this!). Considering in the same way the remaining parts of the convex polygon that are outside the regular hexagon, we conclude that the area of the part of convex polygon outside of hexagon is not greater than the triple

Figure 8.3.

area of the triangle with the side equal to the side of hexagon, i.e. it is not greater than a half of the area of the regular hexagon. This proves the desired inequality.

The equality holds if and only if the vertices of the convex polygon lie on sides of the starshaped region on the picture, i.e. if and only if the convex polygon is a triangle (prove this!).

8.4. Supose by contradiction that such bijection does not exist. For $m = n = 1$ we get $f(1) + 3f(1)^2 = 0$, i.e. $f(1) = 0$. By bijectivity of f it follows $f(n) \geq 1$ for all $n \geq 2$. If $m, n \geq 2$, then $f(mn) \geq 1 + 1 + 3 = 5$, so that $f(k) \geq 2$ for every nonprime integer k. Therefore there exist different primes n_1 and n_3 such that $f(n_1) = 1$, $f(n_3) = 3$ and an integer n_8 such that $f(n_8) = 8$. Then $f(n_3^2) = 3 + 3 + 3 \cdot 3 \cdot 3 = 33$ and $f(n_1 n_8) = 1 + 8 + 3 \cdot 1 \cdot 8 = 33$. From this we conclude $n_3^2 = n_1 \cdot n_8$, i.e. $n_1 \,|\, n_3^2$, which is impossible, because n_1 and n_3 are two different primes.

Remark. If we denote $g(m) = 3f(m) + 1$, the property of f is translated into $g(mn) = g(m)g(n)$, $g\colon \mathbf{N} \to 3\mathbf{N}_0 + 1$, hence the monoids $(\mathbf{N}, \cdot)$ and $(3\mathbf{N}_0 + 1, \cdot)$ are not isomorphic.

Figure 1

SOLUTIONS OF PROPOSALS

1. Let the points of the plain be 3–coloured. Consider the configuration (see Figure 1) in which all lines have a unit length.

Let A have, say, red colour. Then points B and C must be blue and green (in some order). Therefore F must be red. Similarly G must be red. But then F and G form a pair at distance 1 with the same colour.

2. First we shall solve a similar problem for a collection S of subsets of $\mathbf{N}\times\mathbf{N}$, no one containing another. Let S be a collection of subsets A_n of $\mathbf{N}\times\mathbf{N}$, defined by

$$A_n = \{(p,q)\colon p \neq n \text{ and } q \geq n\},$$

for each $n \in \mathbf{N}$.

Note that if $m \neq n$, then $A_m \not\subseteq A_n$. Suppose that B intersects every A_n. Then B must be infinite. If there exists n_0 such that a set

$$\{q\colon (n_0,q) \in B\}$$

is infinite, then let $(n_0,q) \in B$ and let $B' = B \setminus \{(n_0,q)\}$. If there is no such n_0, let (p,q) be arbitrary and let $B' = B \setminus \{(p,q)\}$. In either case B' still intersects every A_n. This shows that $C(S)$ must be empty.

Figure 4

Let $f\colon \mathbf{N}\times\mathbf{N}\to\mathbf{N}$ be a bijection such that $f(1990,1989)=1989$. Let

$$\mathcal{F}=\{f(A_n)\colon A_n\in S\}.$$

The collection $\mathcal{F}$ is our example.

3. There exist natural numbers x so that $n_x\neq 0$; let m be the biggest and F a face with m vertices. There are m distinct faces F_i adjecent to F. The number v_i of F_i's vertices satisfies $3\leq v_i\leq m$, hence there are at most $(m-2)$ values k for v_i.

If there exists k so that at least three faces F_i have k vertices, (i) is true.

If (i) is not true, there will be at least two distinct values k and h such that two faces F_i have k vertices and two have h vertices; in this situation (ii) is true.

4. We shall construct the desired n numbers by repeating a phase of temporary selections. In the first phase we choose the least element in each row, and then for each column we choose the biggest of the numbers selected (if any). If there is one selected number in each column, we are finished, otherwise there are columns for which some of the row selected numbers are rejected. We shall call these rows free and we pass to the next phase.

For any subsequent phase we choose for each free row the least element fromm those not yet tried. If each column contains a selected number, we are finished, otherwise we consider the biggest number in each column (if any), call rejected rows free, and go to the next phase.

Let us note that an arbitrary row can be rejected at most $n-1$ times, hence in the worst case we can have at most $n(n-1)+1$ phases. Thus the construction just described is finite.

The final selection of the n numbers has the desired property. Indeed, if a_{ii} is the number selected in row i and a_{ik} is such that $a_{ik} < a_{ii}$, then in our construction we must have a phase in which the selected number in row was a_{ik}. However, this number was rejected during the construction by choosing at least another number in column k bigger than a_{ik}. Hence the final selected number in column k is bigger than a_{ik}.

5. From the recurrence relation we have

$$P_{n+1}(x) = P_n(x)\frac{P_n^2(x)+P_{n-1}^2(x)}{1+P_n(x)P_{n-1}(x)} - P_{n-1}(x),$$

i.e.

$$\frac{P_{n+1}(x)+P_{n-1}(x)}{P_n(x)} = \frac{P_n^2(x)+P_{n-1}^2(x)}{1+P_n(x)P_{n-1}(x)}. \tag{1}$$

On the other hand

$$\frac{P_n^2(x)+P_{n-1}^2(x)}{1+P_n(x)P_{n-1}(x)} = \frac{\left(\frac{P_{n-1}^3(x)-P_{n-2}(x)}{1+P_{n-1}(x)P_{n-2}(x)}\right)^2 + P_{n-1}^2(x)}{1+\frac{P_{n-1}^3(x)-P_{n-2}(x)}{1+P_{n-1}(x)P_{n-2}(x)}P_{n-1}(x)} =$$

$$= \frac{\big(P_{n-1}^3(x)-P_{n-2}(x)\big)^2 + P_{n-1}^2(x)\big(1+P_{n-1}(x)P_{n-2}(x)\big)^2}{\big(1+P_{n-1}(x)P_{n-2}(x)\big)^2 + \big(P_{n-1}^3(x)-P_{n-2}(x)\big)P_{n-1}(x)\big(1+P_{n-1}(x)P_{n-2}(x)\big)}$$

$$= \frac{\big(1+P_{n-1}^4(x)\big)\big(P_{n-1}^2(x)+P_{n-2}^2(x)\big)}{\big(1+P_{n-1}^4(x)\big)\big(1+P_{n-1}(x)P_{n-2}(x)\big)}$$

$$= \frac{P_{n-1}^2(x)+P_{n-2}^2(x)}{1+P_{n-1}(x)P_{n-2}(x)}.$$

Thus we have

$$\frac{P_n^2(x)+P_{n-1}^2(x)}{1+P_n(x)P_{n-1}(x)} = \cdots = \frac{P_2^2(x)+P_1^2(x)}{1+P_2(x)P_1(x)} =$$

$$= \frac{x^6+x^2}{1+x^4} = x^2.$$

Finally from (1) we get

$$P_{n+1}(x) + P_{n-1}(x) = x^2 P_n(x),$$

and our assertion follows by an easy induction.

REMARK This problem is related to the problem no. 6 from the 1988 IMO held in Australia.

6. From the first equation we deduce

$$8 = 2^{x^2+y} + 2^{x+y^2} \geq 2\sqrt{2^{x^2+y} \cdot 2^{x+y^2}},$$

or equivalently

$$4 \geq x^2 + x + y + y^2.$$

Since $(x+y)^2 \leq 2(x^2+y^2)$, we get further

$$(x+y)^2 + 2(x+y) - 8 \leq 0.$$

From this relation we deduce that

$$x + y \leq -1 + \sqrt{9} = 2.$$

On the other hand from $x, y \geq 0$ and from

$$(\sqrt{x} + \sqrt{y})^2 \leq 2(x+y)$$

it follows, using the second equation, that

$$x + y \geq 2$$

and therefore $x + y = 2$.
Now after squaring the second equation and using $x + y = 2$ we deduce that $\sqrt{xy} = 1$, i.e. $xy = 1$. This together with $x + y = 2$ implies $x = y = 1$.

7. We shall prove that $a_n > a_{n+1}$ inductively. For $n = 1$ the statement is trivial.

Assume that for a given n we have $a_{n-1} > a_n$, and let us prove that $a_n > a_{n+1}$. Since

$$a_n = \frac{a_{n-1}(n-1)^{k+1} + n^k}{n^{k+1}} > \frac{a_n(n-1)^{k+1} + n^k}{n^{k+1}},$$

we have

$$(*) \qquad a_n\big(n^{k+1} - (n-1)^{k+1}\big) > n^k.$$

Since

$$a_{n+1} = \frac{a_n n^{k+1} + (n+1)^k}{(n+1)^{k+1}},$$

we must show that

$$\big((n+1)^{k+1} - n^{k+1}\big)a_n > (n+1)^k.$$

From (∗) we conclude it is sufficient to prove

$$\frac{n^k}{n^{k+1} - (n-1)^{k+1}} \geq \frac{(n+1)^k}{(n+1)^{k+1} - n^{k+1}}.$$

This may be written equivalently as

$$\frac{n^{k+1} - (n-1)^{k+1}}{n^k} \leq \frac{(n+1)^{k+1} - n^{k+1}}{(n+1)^k},$$

or as

$$n\left(1 - (\frac{n-1}{n})^{k+1}\right) \leq (n+1)\left(1 - (\frac{n}{n+1})^{k+1}\right),$$

or as

$$n\left((\frac{n}{n+1})^{k+1} - (\frac{n-1}{n})^{k+1}\right) \leq 1 - (\frac{n}{n+1})^{k+1},$$

or as

$$n^{2k+2} - (n^2-1)^{k+1} \leq n^k\big((n+1)^{k+1} - n^{k+1}\big),$$

or as

$$(k+1)n^{2k} - \binom{k+1}{2}n^{2k-2} + \cdots \leq n^k\big((k+1)n^k + \binom{k+1}{2}n^{k-1} + \ldots\big),$$

which is obviously true.

8. By setting $x = 0$ and then $y = 0$ in the relation, we get

$$f(f(y)) = y + f(0), \quad f(x + f(0)) = f(x). \tag{1}$$

Now

$$f(x) = f(x + f(0)) = f(f(f(x))) = f(x) + f(0),$$

and hence $f(0) = 0$. Thus from (1) we have $f(f(y)) = y$. Now

$$f(x+y) = f\big(x + f(f(y))\big) = f(x) + f(y)$$

and the continuity of f implies that $f(x) = ax$ for some $a \in \mathbf{R}$. From the relation in the problem we get

$$a(x + ay) = ax + y$$

Figure 9

for all $x, y \in \mathbf{R}$, i.e. $a = \pm 1$. Therefore

$$f(x) = x \quad \text{or} \quad f(x) = -x.$$

9. Let us denote by M the point where the line EF meets the line AB. Remark that by construction M belongs to the segment AB. We shall show that M is a fixed point, namely

$$\frac{MB}{MA} = \frac{BN}{AB}. \tag{1}$$

where N is the point where the line CE meets the line AB.

Using Menelaus theorem in the triangle ABD, applied for the transversal EF, we get

$$\frac{MB}{MA} \cdot \frac{FA}{FD} \cdot \frac{ED}{EB} = 1,$$

ie.

$$\frac{MB}{MA} = \frac{FD}{FA} \cdot \frac{EB}{ED}. \tag{2}$$

Since CD is parallel to AB, then we have that

$$\triangle AFB \sim \triangle DFC, \quad \triangle ECD \sim \triangle ENB,$$

and therefore

$$\frac{FD}{FA} = \frac{CD}{AB}, \quad \frac{EB}{ED} = \frac{BN}{CD}.$$

Substituting this relations in (2) we get (1).

10. Let E be a point on the line BC such that $AE = AB$ and denote by AH the altitude in the isosceles triangle BEA. We consider the case when E is between B and C (if B is between E and C or $AH \perp BC$, the reasoning is similar). Since

$$AC^2 - CH^2 = AB^2 - BH^2,$$

Figure 10

we have

$$AC^2 - AB^2 = CH^2 - BH^2 =$$
$$= (CH - BH)(CH + BH) = (CH - EH)BC =$$
$$= CE \cdot BC.$$

Now the equality $BC = 2AC - 2AB$ implies

$$2AB + \frac{1}{2}BC = 2CE. \tag{1}$$

a) If $\angle ABD = 2\angle ADB$, then $\angle AEB = 2\angle ADE$ and the triangle ADE is isosceles, so that $AB = AE = DE$. Hence (1) shows that $2DE + \frac{1}{2}BC = 2CE$, which is equivalent to $BD = 3CD$.
b) Conversely, from $BD = 3CD$ it follows that $2DE + \frac{1}{2}BC = 2CE$, and (1) implies $AB = DE$. Since $AB = AE$, we have $AE = DE$. Hence $\angle ABD = 2\angle ADB$.

11. a) Assume the inequality in the problem holds. If $A = B$, then $PA \geq PC$ for each P and so $A = C$.

Let $A \neq B$. First we shall prove that C lies on the line AB. Suppose by contradiction that $C \notin AB$ and let P be a point from the perpendicular bisector of the line–segment AB such that P and C are from different sides of the perpendicular bisector of AC. Then $PA = PB$, $PA < PC$, so that

$$PA^{1989} + PB^{1989} = 2PA^{1989} < 2PC^{1989},$$

which is a contradiction. Therefore C lies on the line AB.

Denote by O the midpoint of the line–segment AB. Without loss of generality we may assume that O is between B and C. Consider a point $P \in AB$ on the right of B and let $PO = x$, $OC = c$, $AO = BO = a$. Then by the inequality in the problem it follows that

$$f(x) = (x-a)^{1989} + (x+a)^{1989} - 2(x+c)^{1989} \geq 0$$

for all $x \geq 0$. It is easy to see that $f(x)$ is a polynomial whose free term is equal to $-2c^{1989}$. Since $c \geq 0$ and $f(x) \geq 0$ for all $x \geq 0$, it follows that $c = 0$, i.e. C is the midpoint of AB.

b) Conversly, let C be the midpoint of AB. Then

$$\frac{PA + PB}{2} \geq PC$$

for every point P, and from the well known inequality

$$\frac{x^n + y^n}{2} \geq \left(\frac{x+y}{2}\right)^n, \quad \forall x, y \geq 0$$

we obtain

$$\frac{PA^{1989} + PB^{1989}}{2} \geq \left(\frac{PA + PB}{2}\right)^{1989} \geq PC^{1989},$$

which was to be proved.

The assertion remains true if 1989 is replaced by an arbitrary natural number n.

12. Let X be an arbitrary point in $ABCD$. Denote by M, N, P, Q the orthogonal projections of X on the faces BCD, CAD, ABD, ABC respectively, and let

$$XM = x, \quad XN = y, \quad XP = z, \quad XQ = t.$$

Since XM, XN and XP are pairwise perpendicular, we have

$$V_{MNPX} = \frac{1}{6}xyz.$$

Figure 12

It can be easily verified that

$$V_{PQMX} = \frac{1}{6} xzt \sin\alpha,$$

where α is the angle between the line XQ and its orthogonal projection on the plane (XPM). Since the planes (XPM) and (ACD) are parallel, α is the angle between the altitude of the pyramid from the vertex D and one of the side faces. We may assume that $AD = BD = CD = 1$. Then $AB = BC = CA = \sqrt{2}$ and $\sin\alpha = \sqrt{3}/3$. So

$$V_{PQMX} = \frac{1}{6}\frac{\sqrt{3}}{3} xzt, \quad V_{QMNX} = \frac{1}{6}\frac{\sqrt{3}}{3} xyt,$$

etc. Therefore

$$V(X) = V_{MNPQ} = \frac{1}{6}\big(xyz + \frac{\sqrt{3}}{3} t(xy + yz + zx)\big).$$

Let x', y', z' be distances from Q to the faces BCD, CAD and ABD of the pyramid. Then

$$\begin{aligned} x' &= x + t\sin\alpha = x + \frac{t}{\sqrt{3}}, \\ y' &= y + \frac{t}{\sqrt{3}}, \\ z' &= z + \frac{t}{\sqrt{3}}. \end{aligned}$$

Now we obtain

$$\begin{aligned} V(Q) &= \frac{1}{6}\big(x + \frac{t}{\sqrt{3}}\big)\big(y + \frac{t}{\sqrt{3}}\big)\big(z + \frac{t}{\sqrt{3}}\big) \\ &\geq \frac{1}{6}\big(xyz + \frac{\sqrt{3}}{3}t(xy + yz + zx)\big) = V(X), \end{aligned}$$

and equality occurs only for $t = 0$, i.e. for $X = Q$. Hence we may consider that X lies in the triangle ABC. In this case $t = 0$ and

$$\frac{1}{6} = V_{ABCD} = \frac{1}{6}(x + y + z),$$

i.e.

$$x + y + z = 1.$$

Since $V(X) = \frac{1}{6}xyz$, we obtain by the arithmetic mean–geomteric mean inequality that $V(X)$ is maximal if $x = y = z = \frac{1}{3}$, i.e. when X is the barycenter of the triangle ABC.

13. Let $A_1, A_2, \ldots, A_n$ be vertices of the n–gon, α_i the angle at the vertix A_i,

$$a_1 = |A_1A_2|,\ a_2 = |A_2A_3|, \ldots, a_n = |A_nA_1|,$$

and S_i the area of the triangle $A_{i-1}A_iA_{i+1}$, $i = 1, 2, \ldots n$ ($A_0 = A_n$, $A_{n+1} = A_1$). Then we have

$$2S_i = a_{i-1}a_i \sin\alpha_i.$$

Let $S = \min(S_1, S_2, \ldots, S_n)$. We have

$$2S \leq a_{i-1}a_i \sin\alpha_i \quad (i = 1, 2, \ldots, n)$$

and

$$(2S)^n \leq \prod_{i=1}^{n} a_i^2 \prod_{i=1}^{n} \sin a_i \leq \prod_{i=1}^{n} a_i^2.$$

Figure 13

Using the inequality

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} a_i}{n},$$

we obtain

$$2S \leq \left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^2.$$

If p_i and q_i are lengths of the projections of a_i over the sides of the square, then

$$a_i \leq p_i + q_i$$

i.e.

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i \leq 4.$$

Therefore

$$S \leq \frac{8}{n^2}.$$

Alternate solution. Denote by p_i' and q_i' the corresponding lengths of projections of $A_{i-1}A_{i+1}$. Retaining the notation from the preceding solution, we have as a direct consequence of the convexity of n–gon the following inequality:

$$\sum_{i=1}^{n} p_i' \leq \sum_{i=1}^{n} (p_{i-1} + p_i) = 2\sum_{i=1}^{n} p_i \leq 4,$$

and analogously $\sum_{i=1}^{n} q_i' \le 4$. Then from

$$(p_1' + q_1') + (p_2' + q_2') + \cdots + (p_n' + q_n') \le 8,$$

using the Dirichlet principle we see that there is i such that $p_i' + q_i' \le 8/n$. So, for the area of the corresponding triangle we have

$$S_i \le \frac{1}{2} p_i' q_i' \le \frac{1}{2} p_i' (\frac{8}{n} - p_i') \le \frac{8}{n^2}.$$

14. Let $n_1 = n^{n^{n^n}} - n^{n^n}$ and $n_2 = n^{n^n} - n^n$. Then

$$n_1 = n^{n^n}(n^{n_2} - 1).$$

We shall show that

(1) $$3 \cdot 2^4 \,|\, n_2.$$

Observe that $n_2 = n^n(n^{n^n - n} - 1)$ and $n^n - n$ is even, thus $\varphi(3) \,|\, n^n - n$, hence

(2) $$3 \,|\, n_2.$$

If n is even, then $n^n \,|\, n_2$, thus

(3') $$2^n \,|\, n_2.$$

and $2^4 | n_2$, because $n \ge 3$. If n is odd, $n = 2k + 1$, then

$$\begin{aligned} n^n - n &= (2k+1)\big((2k+1)^{2k} - 1\big) = \\ &= (2k+1)\big((4k(k+1)+1)^k - 1\big) = \\ &= (2k+1)\big((8l+1)^k - 1\big) = 8p. \end{aligned}$$

Thus $8 \,|\, n^n - n$. Since $\varphi(2^4) \,|\, n^n - n$, we have

(3") $$2^4 \,|\, n^{n^2 - n} - 1.$$

From (2), (3') and (3") we have (1).

From (1) we obtain $\varphi(3^2) = 6 \,|\, n_2$, thus

(4) $$3^2 \,|\, n^2(n^{n_2} - 1).$$

Similarly $\varphi(13) \,|\, n_2$ and $\varphi(17) \,|\, n_2$, so that

(5) $$13 \,|\, n(n^{n_2} - 1)$$

(6) $$17 \,|\, n(n^{n_2} - 1).$$

From (4), (5) and (6) we conclude that $1989 = 3^2 \cdot 13 \cdot 17 \mid n_1$.

15. Note that if $a_k \equiv 0 \pmod{11}$, then

$$\begin{aligned} a_{k+1} &\equiv 0^2 - 0 + 5 \equiv 5 \pmod{11}, \\ a_{k+2} &\equiv 5^2 - 5 + 5 \equiv 3 \pmod{11}, \\ a_{k+3} &\equiv 3^2 - 3 + 5 \equiv 0 \pmod{11}. \end{aligned}$$

Therefore it suffices to prove that at least one of the numbers

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5$$

is divisible by 11. We can check directly that:

$$\begin{array}{lllll} \text{if} & a_0 \equiv 1,\, 2,\, 10 & \pmod{11} & \text{then} & a_3 \equiv 0 \pmod{11}, \\ \text{if} & a_0 \equiv 3,\, 9 & \pmod{11} & \text{then} & a_1 \equiv 0 \pmod{11}, \\ \text{if} & a_0 \equiv 4,\, 8 & \pmod{11} & \text{then} & a_5 \equiv 0 \pmod{11}, \\ \text{if} & a_0 \equiv 5,\, 7 & \pmod{11} & \text{then} & a_2 \equiv 0 \pmod{11}, \\ \text{if} & a_0 \equiv 6 & \pmod{11} & \text{then} & a_4 \equiv 0 \pmod{11}. \end{array}$$

This completes the solution.

16. Let $x_i = (a_i, S_i)$ and $y_i = [a_i, S_i]$. Then $x_i y_i = a_i S_i$ $(i = 3, 4, \ldots, 1989)$ and the given relation can be written in the form

$$a_i + S_i = x_i + \frac{a_i S_i}{x_i},$$

i.e. $x_i^2 - (a_i + S_i)x_i + a_i S_i = 0$. So if $a_i \mid S_i$, then $x_i = a_i$ and the equation holds true.

Thus we may take $a_1 = 1$, $a_2 = 2$ and $a_i = 2^{i-3} \cdot 3$, $i \geq 3$, whereupon we have $S_i = 3 \cdot 2^{i-2}$, $i = 3, 4, \ldots, 1989$.

17. Let $x \leq y \leq z$, n be one solution. It is obvious that $z - x \geq y - x \geq 0$ and $z + x > y > 0$. Hence $(z - x)(z + x) \geq y(y - x)$, i.e.

$$z^2 \geq x^2 + y^2 - xy = \frac{x^3 + y^3}{x + y},$$

or

$$x + y \geq \frac{x^3 + y^3}{z^2}.$$

Then it is easy to see that

$$z = nx^2y^2 - \frac{x^3 + y^3}{z^2} \geq nx^2y^2 - (x + y) > 0,$$

except for $x = y = 1$, $n = 1, 2$ (in this case there is no solution to the problem).

But $z^2 \mid x^3 + y^3$, and from the previous inequality we obtain

$$x^3 + y^3 \geq z^2 \geq (nx^2y^2 - (x + y))^2.$$

Therefore

$$n^2x^4y^4 \leq x^3 + y^3 + 2nx^2y^2(x + y) - (x + y)^2$$

i.e.

$$n^2x^4y^4 < 2nx^2y^2(x + y) + x^3 + y^3$$

or

$$nxy < 2(\frac{1}{x} + \frac{1}{y}) + \frac{1}{nx^3} + \frac{1}{ny^3}.$$

If $x \geq 2$, then $y \geq x \geq 2$ and $nxy \geq 4$. But

$$2(\frac{1}{x} + \frac{1}{y}) + \frac{1}{nx^3} + \frac{1}{ny^3} \leq 2(\frac{1}{2} + \frac{1}{2}) + \frac{1}{8} + \frac{1}{8} < 3,$$

which is a contradiction. Hence $x = 1$ and we have

$$ny < 2 + \frac{2}{y} + \frac{1}{n} + \frac{1}{ny^3}.$$

If $y \geq 4$, then $ny \geq 4$ and

$$2 + \frac{2}{y} + \frac{1}{n} + \frac{1}{ny^3} < 2 + \frac{2}{4} + 1 + \frac{1}{4} < 4,$$

which is impossible. Therefore $y \leq 3$. Since $z^2 \mid x^3 + y^3$, i.e. $z^2 \mid 1 + y^3$ and $z \geq y$, we have several possibilities

(i) if $y = 1$, then $1 + y^3 = 2$ and $z = 1$;
(ii) if $y = 2$, then $1 + y^3 = 9$ and $z = 3$;
(iii) if $y = 3$, then $1 + y^3 = 28$ and z does not exist.

If $x = y = z = 1$, then $n = 3$. If $x = 1$, $y = 2$, $z = 3$, then $n = 1$.

So all solutions of the problem are

$$(1, 1, 1, 3) \quad (1, 2, 3, 1) \quad (2, 1, 3, 1) \quad (1, 3, 2, 1)$$
$$(3, 1, 2, 1) \quad (3, 2, 1, 1) \quad (2, 3, 1, 1).$$

18. Let z be a common root of $x^5 - px - 1$ and $x^2 - ax + b$. If z is rational, since $z^5 - pz - 1 = 0$, then $z = \pm 1$. Hence $p = 0$ or $p = 2$. It is easy to be seen that $p = 0$ or $p = 2$ satisfy the condition of the problem.

Suppose z is not rational. Then from

$$\begin{aligned} pz+1 = z^5 &= z(az-b)^2 = z(a^2z^2-2abz+b^2) = \\ &= z(a^2(az-b)-2abz+b^2) = (a^3-2ab)z^2+(b^2-a^2b)z = \\ &= (a^3-2ab)(az-b)+(b^2-a^2b)z = \\ &= (a^4-3a^2b+b^2)z+2ab^2-a^3b. \end{aligned}$$

we conclude

$$\begin{aligned} a^4-3a^2b+b^2 &= p \\ 2ab^2-a^3b &= 1. \end{aligned}$$

We multiply the first equation by $-2a$ and add to the second. Then

$$b = \frac{2a^5-2ap+1}{5a^3},$$

and substituting this in the second equation, after some computation we get

$$a^{10}+3pa^6+11a^5-4p^2a^2+4pa-1=0.$$

Since a is rational and p is an integer, we have $a=\pm 1$.

Let $a=1$. Then $-4p^2+7p+11=0$ and p is not an integer.

Let $a=-1$. Then $-4p^2-p-11=0$, and p is not an integer either.

So $p=0$ and $p=2$ are the only solutions.